Theoretical Models of Suppression of Instabilities in Hall Thruster by Shear of Magnetic Field

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Abstract: In the paper, influences of the magnetic field shear on the instability in the near-anode region of the conventional Hall thruster (HT) and on the instability in the near-anode cavity of the CAMILA HT are theoretically studied. These instabilities can lead to the unfavorable potential distribution in areas of their localization. It is shown that the acceptable changes of the magnetic field lines direction should result in significant reducing the growth rate of the instabilities.

I. Introduction

One of the major problems faced by the developers of Hall thrusters is the optimization of the distribution of the electric field in the acceleration channel in order to ensure the maximal efficiency and required lifetime of the thruster. Taking into account the fact that the distribution of the electric field in the acceleration channel is largely determined by the amplitudes of some plasma perturbations in the thruster, the problem of controlling the instabilities that excite these perturbations is of considerable interest. Among the instabilities which affect strongly the distribution of the electric field, the next two occupy an important place: 1) relatively low-frequency instability in the near-anode region of conventional Hall thrusters, where the drift velocity due to a finite temperature of electrons and a non-uniformity of the magnetic field can exceed the electrical drift velocity, and 2) relatively low-frequency instability in the anode cavity of the CAMILA Hall thruster.

The near-anode region of the conventional Hall thruster, defined as the area of the acceleration channel between the anode and the zone of intense ionization, occupies a significant fraction of the channel – up to half of its length. Even for this reason alone, the near-anode region needs a caring attitude towards it. More specifically, the near-anode area performs a function of great importance: it provides transfer of electrons, born in the zone of ionization, to the anode. In the near-anode region, the electrons should move to the anode without exiting oscillations which can influence on stability of the discharge firing in the whole acceleration channel.

In the anode cavity of the CAMILA Hall thruster (Fig.2), the radial electrical field should prevent ion colliding the walls of the anode cavity. It is possible at the large enough resistance of plasma with magnetized electrons. However, the real conductivity across the magnetic field is Bohm’s one that significantly reduces barrier’s electrical field for ions. The reason of this is a strong instability due mainly to the positive density gradient along the electrical field.

Thus, we need suppression of noted above instabilities. The application of a well-established method to suppress these instabilities, namely, using the positive gradient of the magnetic field is unacceptable since the positive gradient of the magnetic field along the acceleration channel is itself the reason of the instability in the first case and it is difficult to create it in the anode cavity in the second case.

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II. Theoretical Model of Low-Frequency Instability in Near-Anode Region with Taking into Account Motion of Perturbed Electrons Along Magnetic Field

A. Basic Assumptions and Governing Equations

Usually at a study of hydrodynamic instabilities due to convection of charged particles the perturbation wavelength is explicitly or implicitly assumed to be equal to infinity along the magnetic field, that is, \( k_z = 0 \). Now, we discard this assumption.

We use the assumptions which were applied in Ref. 1, namely: 1) Cartesian frame with X and Z axes directed parallel to the thruster axis and the applied magnetic field, respectively, as shown in Fig.2; 2) the perturbations are assumed to be potential, quasi-neutral and non-dissipative; 3) the ionization processes are not taken into consideration; 4) the analysis of the stability of the plasma is carried out under the approximation of two-fluid magneto-hydrodynamics with cold non-magnetized ions and hot magnetized electrons; 5) the temperature of the electrons is assumed to be constant (the finite temperature of the electrons is to be taken into account because in the near-anode region, as distinguished from the acceleration area, the velocity of the drift motion of the electrons, due to the temperature of the electrons, is comparable in magnitude with the velocity of the electrical drift).

The finite temperature of the electrons brings about two drifts: 1) Larmor drift due to the gradient of electron pressure; 2) drift due to the gradient of the magnetic field. The latter deserves special attention. If a magnetized charge particle moves in a curvilinear magnetic field (any non-uniform magnetic field in the absence of the electrical current is a curvilinear one), it experiences a force. This force is directed to the convexity of the magnetic field lines and transverse to the magnetic field. (See, for example, Ref. 10) For electrons, the expression for the force is as follows:

\[
|F_B| \approx \frac{mv^2}{R} = \frac{2T_e}{R} |\nabla \ln B_0| 
\]  

(1)
where $R$ – the radius of curvature of the magnetic field line, $B_0$ – the module of induction of the magnetic field.

(We assume that the pressure of the plasma is much less than the pressure of the applied magnetic field.)

Under the effect of the force, the electron drifts transverse to both the magnetic field and the force.

Figure 2. A sketch of near-anode region in conventional Hall thruster

As a distinction from Ref. 1, we take into consideration moving the perturbed electrons along the magnetic field. In doing so, we assume that $\omega/kz > v_{Te}$ (2), where $v_{Te}$ is the thermic velocity of electrons.

Under the above assumptions, the linearized set of equations of the two-fluid magneto-hydrodynamic model in the small-scale approximation for the perturbations takes the following form:

Equations for the ion component

$$\frac{\partial V_x}{\partial t} + V_{x0} \frac{\partial V_x}{\partial x} = - \frac{e}{M} \frac{\partial \Phi}{\partial x}$$  \hspace{1cm} (3)

$$\frac{\partial V_y}{\partial t} + V_{y0} \frac{\partial V_y}{\partial x} = - \frac{e}{M} \frac{\partial \Phi}{\partial y}$$  \hspace{1cm} (4)

$$\frac{\partial n}{\partial t} + n_0 \frac{\partial V_x}{\partial x} + V_{x0} \frac{\partial n}{\partial x} + n_0 \frac{\partial V_y}{\partial y} = 0$$  \hspace{1cm} (5)
Equations for the electron component

\[ e \frac{\partial \Phi}{\partial x} + \frac{T_{e0}}{n_0^2} \frac{\partial n_0}{\partial x} n - \frac{T_{e0}}{n_0} \frac{\partial n}{\partial x} - eu_y B_0 = 0 \]  
\[ (6) \]

\[ e \frac{\partial \Phi}{\partial y} - \frac{T_{e0}}{n_0} \frac{\partial n}{\partial y} + eu_x B_0 = 0 \]  
\[ (7) \]

\[ \frac{\partial u_z}{\partial t} + u_{x0} \frac{\partial u_z}{\partial x} + u_{y0} \frac{\partial u_z}{\partial y} + \frac{e}{m} \frac{\partial \Phi}{\partial z} = 0 \]  
\[ (8) \]

\[ \frac{\partial n}{\partial t} + n_0 \frac{\partial u_x}{\partial x} + n_0 \frac{\partial u_y}{\partial y} + n_0 \frac{\partial u_z}{\partial z} = 0 \]  
\[ (9) \]

where \( V_x, V_y \) – the projections of perturbed velocity of the ions on the \( X \) and \( Y \) axis, respectively, \( u_x, u_y, u_z \) – the projections of perturbed velocity of the electrons on the \( X, Y \) and \( Z \) axis, respectively, \( M \) – the ion mass, \( m \) – the electron mass, \( \Phi \) – the plasma potential perturbation.

The quantities with index “0” are non-perturbed.

The non-perturbed electrical field \( E_{x0} \) is directed towards the anode that due to the large gradient of the electron pressure in the near-anode region of the conventional Hall thruster. The non-perturbed velocity of the electrons \( u_{y0} \) is directed along the \( Y \) axis. To find it, we use the projection of the equation of non-perturbed motion of electrons on the \( X \) axis:

\[ -en_0 E_{x0} - \frac{2T_{e0}n_0}{R} - \frac{T_{e0}}{n_0} \frac{\partial n_0}{\partial x} - eu_{y0} B_0 n_0 = 0 \]  
\[ (10) \]

From Eq. (9), it follows:

\[ u_{y0} = -\frac{E_{x0}}{B_0} - \frac{2T_{e0}}{eB_0 R} - \frac{T_{e0}}{eB_0 n_0} \frac{\partial n_0}{\partial x} \]  
\[ (11) \]

Below, the approximate equality \( \frac{1}{R} \approx \frac{1}{B_0} \frac{\partial B_0}{\partial x} \) is used.

**B. Dispersion Equation**

We seek the solution of the set of equations (3) – (9) and Eq. (10) in the form:

\[ \mathbf{F}(x, y; t) = \mathbf{F}_k \exp(-i(\omega t - k_x x - k_y y - k_z z)) \]

where \( \mathbf{F} \) and \( \mathbf{F}_k \) are consequently the vector of perturbed parameters and vector of their Fourier components, \( k_x, k_y \) and \( k_z \) the projections of the wave vector on the axes \( X, Y \) and \( Z \), respectively.

In doing so, we restrict ourselves for simplicity by the case \( k_x V_{y0} \ll \omega \ll k_z u_{y0} \)  
\[ (12) \]
As a result, we obtain the following local dispersion equation,

$$\omega^2 = \frac{e(k_x^2 + k_y^2) \left( E_{x0} + \frac{2T_{e0}}{e I_B} \right)}{M \left( \frac{1}{l_B} - \frac{1}{l_N} \right)} = 0$$

(13)

$$\gamma = \frac{e(k_x^2 + k_y^2) \left( \frac{2T_{e0}}{e I_B} - |E_{x0}| \right)}{M \left( \frac{1}{l_N} - \frac{1}{l_B} \right)}$$

(14)

From the dispersion equation (13), it follows that at \( 0 < l_N < l_N \) and \( |E_{x0}| < 2T_{e0}/e I_B \), we have aperiodic instability with the growth rate of

One can see that taking into account \( k_z \neq 0 \) leads to reducing the growth rate of the instability. At first glance, at the width of the acceleration channel of \( h = 1.5 \cdot 10^{-2} \text{ m} \), a minimal value of \( k_z = \frac{\pi}{h} \) should be equal to \( 3.14 \cdot 10^2 \text{ m}^{-1} \). Then, at \( B_0 = 10^2 \text{ T} \), \( T_{e0} = 10 \text{ eV} \), \( l_N = 9 \cdot 10^{-3} \text{ m} \), \( l_B = 10^{-2} \text{ m} \), \( E_{x0} = -1.8 \cdot 10^3 \text{ V/m} \), \( k_x \approx k_x = 2 \cdot 10^2 \text{ m}^{-1} \), the growth rate of the instability should be reduced by a factor of 23.4, that is, should be negligible small. However, this is not the case. The fact is that at frequencies well below than electron Langmuir one, the potential of plasma in Hall thruster does not almost change along a length of the magnetic field line (a phase shift is almost absent), excepting an abrupt change near the wall of the thruster. This justifies application of \( k_z = 0 \) approximation in theoretical models of low-frequency perturbations in the conventional magnetic field (without shear!).

III. Estimation of Influence of Magnetic Field Shear on Low-Frequency Instability in Near-Anode Region

Now let the magnetic induction vector \( \mathbf{B}_0 \) have two components \( B_{z0} \) and \( B_{y0} \), which depend on \( x \). The latter simulates the azimuthal component of the magnetic field, which should be applied to create the sheared magnetic field (Fig. 3).

$$\mathbf{B}_0 = B_{z0}(x) \mathbf{e}_z + B_{y0}(x) \mathbf{e}_y$$

(15)
Figure 3. Schematic of magnetic field shear in Hall thruster

(For simplicity, the magnetic field lines are assumed to be straight ones.) Divided both left and right parts of Eq.(15) into $B_0$, we obtain:

$$\mathbf{e}_0 = h_z(x) \mathbf{e}_z + h_y(x) \mathbf{e}_y$$

Where $\mathbf{e}_0$ is the unit vector in the direction of the magnetic field line,

$$h_z = \frac{B_{z0}}{B_0}, \quad h_y = \frac{B_{y0}}{B_0}$$

We write the equations for the electron component (6) – (9) with a new magnetic field in old Cartesian coordinates, after that convert them to new Cartesian coordinates:

$$x = x', \quad y = h_z y' + h_y z', \quad z = -h_y y' + h_z z'$$

The new Cartesian coordinates correspond to the rotation of the $Y'$ and $Z'$ axes around $X$ axis until $Z'$ coincides with the direction of the new magnetic field line. Performed Fourier-transformation of all perturbed variables and substituted the expressions for the perturbed velocities into the continuity equation, we obtain, with taking into account Eq. (12), the following equation for the electron density:

$$n = -\frac{B_0 \Phi}{E_{x0} + \frac{2T_{e0}}{eB_0} \frac{\partial B_0}{\partial x}} \left[ \frac{1}{k_b} \frac{\partial}{\partial x} \left( \frac{k_n n_0}{B_0} \right) + \frac{ek^2 n_e B_0}{mk_b \left( E_{x0} + \frac{T_{e0}}{eB_0} \frac{2}{\partial x} + \frac{1}{n_0 \partial x} \right) + \frac{2}{\partial x} + \frac{1}{n_0 \partial x} \right]$$

Where

$$k_{b} = k_z h_z - k_y h_y$$

$$k_{||} = k_z h_z + k_y h_y$$

Indexes $b$ and $||$ denote along binormal to the magnetic field and along the magnetic field, respectively.

We assume that $\frac{\omega}{k_{||}} > v_{te}$

In the equation (19) under the sign denoting the derivative there is $k_b$. Though according to Eqs (20), $k_b$ depends on $x$, however at the small deviation of $B_0$ from the initial direction (i.e. at $h_z \approx 1$), the change of $k_b$ is only slight since at all interesting for suppression of instabilities values of $k_z, k_y << k_y$. Therefore, one can take out $k_b$ from under the sign of the derivative. Then, we have:
\[
\frac{n}{E_{x0} + \frac{2T_{e0}}{el_B}} \left[ \frac{1}{l_B} - \frac{1}{l_N} \right] - \frac{eB^2_{||}k^2}{mk^2_{b}\left( E_{x0} + \frac{T_{e0}}{e}\left( \frac{2}{l_B} + \frac{1}{l_N} \right) \right)} \right] \phi = 0
\]

(22)

(It is necessary to note that because of large term \(k_yh_y\) in the second equation of Eqs (20), the small deviation of \(B_0\) from initial direction can lead to the large change of \(k_{||}\).)

After performing the same procedure with the equations (3) – (5), with taking into account Eq. (12), we obtain the equation for the ion density:

\[
n = \frac{e(k^2_{x} + k^2_{y})n_i \Phi}{M\omega^2}
\]

Equating the right sides of equations (19) and (20) gives the following dispersion equation:

\[
\omega^2 - \frac{e(k^2_{x} + k^2_{y}) \left( E_{x0} + \frac{2T_{e0}}{el_B} \right)}{M \left( \frac{1}{l_B} - \frac{1}{l_N} \right)} = 0
\]

(24)

From the dispersion equation, we obtain the growth rate of the instability:

\[
\gamma = \sqrt{\frac{e(k^2_{x} + k^2_{y}) \left( \frac{2T_{e0}}{el_B} - \left| E_{x0} \right| \right)}{M \left( \frac{1}{l_B} - \frac{1}{l_N} \right)} + \frac{k^2_{b}eB^2_{0}}{k^2_{b}m \left( \frac{T_{e0}}{e} \left( \frac{2}{l_B} + \frac{1}{l_N} \right) - \left| E_{x0} \right| \right) \left( \frac{1}{l_B} - \frac{1}{l_N} \right)}}
\]

(25)

If at any point \(x_0\) \(k_{||}(x_0) = k_{x}h_{x}(x_0) + k_{y}h_{y}(x_0) = 0\), then near \(x_0\)

\[
k_{||}(x) = \left[ h^2_{b}k^2_{b} \frac{\partial}{\partial x} \left( \frac{B_{x0}}{B_{z0}} \right) \right]_{x=x_0} (x-x_0)
\]

(26)

(See, for example, Ref. 10)

From Eq. (26) and Eq. (20), it follows that if at some \(k_{x,\max}\), the growth rate of the instability attains zero in the magnetic field without shear, then the shear provides a suppression of the instability at \(\Theta > k_{\max}k_{||}L/k_B\).

(27)

Where \(\Theta \equiv Lh^2_{b} \frac{\partial}{\partial x} \left( \frac{B_{0y}}{B_{0z}} \right)\) has meaning of the angle of the magnetic field line rotation on the distance \(L\), i.e. it is the measure of the rate of the magnetic field line direction change, \(L\) can be considered as the length of the instability localization area.
In our case, at any $k_z$ and, correspondently, at any $k_{\parallel}$, the growth rate does not attain zero. Therefore, we should replace inequality (27) by the following equality

$$\Theta \geq \frac{k_{\parallel} e B_0^2}{k_{\parallel}}$$

(28)

It connects $k_{\parallel} e B_0^2$, at which the growth rate reduces essentially (for example, by a factor of 2), with the angle of the rotation of the magnetic field lines.

Let us consider the sample from section II. From (22), it follows:

$$\gamma_{\theta} = \left(1 + \frac{k^2 B_0^2}{k_{\parallel} m \left(2 T e_0 / l_B - |E_{s0}| \right) \left(1/l_N - 1/l_B \right)} \right)^{-1/2}$$

(29)

Where $\gamma_{\theta}$ and $\gamma_0$ – the growth rate with shear and without shear, respectively.

From Eq. (29), we obtain that at parameters of the near-anode region and plasma in it, presented in section II, $\gamma_{\theta}/\gamma_0 = 1/2$ at $k_{\parallel} e B_0^2/k_{\parallel} = 4.96 \cdot 10^{-2}$. Substituting the obtained ratio of the wave numbers into Eq. (26), we have $\Theta \geq 0.156$.

### IV. Estimation of Influence of Magnetic Field Shear on Convective Instability in Anode Cavity of CAMILA Hall Thruster

In the anode cavity of the CAMILA Hall thruster $T_e e U_a < 1$, where $U_a$ is the needed drop of the potential between the anode and centerline of the cavity providing ionization of Xe-atoms predominantly in the cavity. Therefore, as first approximation, we neglect by the thermic effects. For the rest, we use the same assumptions, differential equations, procedures as under consideration of the influence of the magnetic field shear on the low-frequency instability near-anode region. As a result, we obtain the following dispersion equation:

$$\omega^2 - \frac{e(k_{\parallel}^2 + k_{\parallel}^2)E_{s0}}{M \left(1/l_B - 1/l_N\right)} = 0$$

(30)

As a distinction from the near-anode region of the conventional Hall thruster, in the anode cavity of the CAMILA Hall thruster $E_{s0} > 0$ and $l_B < 0$. In this case, we have the aperiodic instability with the growth rate

$$\gamma = \sqrt{\frac{e(k_{\parallel}^2 + k_{\parallel}^2)E_{s0}}{M \left(1/l_N + 1/l_B\right)} + \frac{k_{\parallel}^2 e B_0^2}{k_{\parallel}^2 m E_{s0} \left(1/l_N + 1/l_B\right)}}$$

(31)

As in the case of the near-anode region of the conventional Hall thruster, in the anode cavity of the CAMILA Hall thruster, the magnetic field shear does not eliminated the instability, but reduces its growth rate.

Let us consider the following sample.
At the distance between co-axial anode and centerline of $6 \times 10^{-3}$ m and the needed drop of the potential of 40 V, we have $E_0 = 6.67 \times 10^3$ V/m. Other parameters: $l_0 = 6 \times 10^{-3}$ m $l_N = 2.4 \times 10^{-3}$ m, $k_x = 5.23 \times 10^2$ m⁻¹, $B_0 = 1.4 \times 10^2$ T. In this case, using Eq. (31) and Eq.(28), we obtain that reducing the growth rate of the convective instability by a factor of 2 can be attained at $\Theta \cong 1$

V. Concluding Remarks

As a result of the conducted investigations, the following it was shown:

1. The shear of the magnetic field influences strongly on the instability growth rates as in the near-anode region of the conventional Hall thruster so in the anode cavity of the CAMILA Hall thruster.

2. In both cases, the shear of the magnetic field does not eliminate instability. It only reduces the growth rate.

3. For reducing the growth of the instability by a factor of 2, it is sufficient to change the direction of the magnetic field line by $\Theta \cong 0.156$ in the near-anode region of the conventional Hall thruster and by $\Theta \cong 1$ in the anode cavity of the CAMILA Hall thruster.

It should be noted the following. Strongly speaking, in the case of the low-frequency instability in the near-anode region of the conventional Hall thruster, inequality (21) is valid at small values of $\Theta$, only. At the large values of the $\Theta$, the kinetic consideration of the problem should be applied.

It is necessary also to note that, as is shown in Ref. 11, imposing the azimuthal magnetic field as an addition to radial magnetic field should not interrupt a drift of electrons in the devices with closed electron drift. In Ref. 11, such the configuration of the magnetic field was suggested to apply for the mass separation of the high density ion flux.

The possible way to create the shear of the magnetic field with in the Hall thruster is using along with the solenoidal winding, the helical winding with opposite directions of a current in neighboring turns.

References


