

Analytic Optimization of Mixed Chemical-Electric Orbit Raising Missions

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This paper gives an analytic derivation of the optimum specific impulse (I_{sp}) for a time and power constrained chemical-electric orbit raising (C-EOR) mission. A derivation of the optimum I_{sp} is made based on a modified form of the rocket equation for a combined C-EOR mission. The result shows that the optimum I_{sp} depends strongly on two parameters: the I_{sp} of the chemical thruster and the mission's trajectory and thrust steering profile. It is also shown that an optimum I_{sp} throttling profile exists for each EOR mission. In one simplified case, use of a constant power, variable I_{sp} thruster during orbit raising increases the mass benefits of EOR by 20% with no power or time penalty. Further work is needed to confirm that this benefit can be achieved with realistic assumptions.

Nomenclature

F	= thrust (N)
I_{sp}^*	= optimum I_{sp} for the electric thruster (sec)
I_{sp}	= specific impulse (sec)
a	= semi-major axis (m)
c_1	= effective chemical thruster exhaust velocity (m/s)
c_2	= effective electric thruster exhaust velocity (m/s)
g	= 9.81 m/s ²
i	= inclination angle (radians)
m_o	= spacecraft mass, beginning of orbit raising (kg)
m_1	= spacecraft mass, end of chemical orbit raising, before EOR (kg)
m_2	= spacecraft mass, end of orbit raising (payload mass) (kg)
\dot{m}_2	= electric propulsion mass flow rate (kg/sec)
P	= thruster input power (W)
t	= time (sec)
v_{ch}	= Characteristic Velocity (m/s)
Δv_{chem}	= Δv for all-chemical orbit raising (m/s)
Δv_1	= Δv for chemical portion of a C-EOR mission (m/s)
Δv_2	= Δv for electric portion of a C-EOR mission (m/s)
Δv_{2eff}	= $\Delta v_{chem} - \Delta v_1$
Δv_{tot}	= $\Delta v_1 + \Delta v_2$
α	= Specific Power (W/kg)

η_p	= Thruster Efficiency
η_v	= Mission Planning Efficiency
θ	= Angle between satellite and line of nodes (radians)
μ	= Gravitational constant * mass of Earth = 3.986 x 10 ¹⁴ m ³ /s ²

Introduction

Western satellites are generally launched into a transfer orbit and then boosted into geostationary orbit using an on board chemical propulsion system. The process is referred to as orbit raising. Recently, commercial manufacturers have expressed interest in the use of electric propulsion systems for orbit raising [1]. Electric orbit raising (EOR) occurs over a period of days or months and results in a net savings of hundreds of kilograms of mass. Although it is possible to accomplish all of orbit raising using only EOR, the time required (many months) is prohibitive for commercial applications. The scenario presently of interest for commercial satellites is a mixed scenario in which chemical propulsion is used for the first part of the mission and electric propulsion is used to complete orbit raising. Combined chemical-electric

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missions (C-EOR) provide customers with significantly more payload mass and yet maintain an acceptable on-orbit delivery time. A typical mission with 90 days of EOR can provide a net dry mass benefit of over 450 kg. per satellite. From the point of view of the satellite owner, optimization of the mission requires that the dry mass benefit per day of EOR, or transportation rate, be maximized. When the power available for EOR is fixed, there is an inverse relationship between thrust and specific impulse. As a result, there is an optimum specific impulse that maximizes this transportation rate.

This paper gives an analytic derivation of the optimum specific impulse for a time and power constrained chemical-electric orbit raising (C-EOR) mission. A derivation is made of a modified form of the rocket equation for the combined C-EOR mission. From this equation, a derivation is made of the optimum specific impulse that maximizes payload mass delivered for a fixed mission duration. The result shows that the optimum I_{sp} is dependent on several parameters typically associated with electric propulsion missions: the length of the mission, the specific power (W/kg), and the total power available. The result also shows that the optimum I_{sp} depends strongly on two additional parameters: the I_{sp} of the chemical thruster and a mission planning efficiency, η_v , which is determined by orbital dynamics and the thrust steering profile. Finally, it is shown that an optimum specific impulse throttling profile exists for a given EOR mission. In some cases, use of a constant power, variable I_{sp} thruster during orbit raising is shown to significantly increase payload mass delivered to orbit without changing the duration of or power required for EOR.

The results provide an easy means to determine the optimum I_{sp} for an EOR mission and to estimate the payload mass benefits. Although the paper is written with an emphasis on orbit raising, the formulation used is general and can be used for any similarly constrained C-EOR mission.

Previous Work

It is well known that for electric propulsion missions, there is an optimum specific impulse (I_{sp}) that maximizes the dry mass delivered at a given specific power (expressed as kg/W). For an all electric mission with a given Δv , this “optimum” I_{sp} can be derived analytically [2] [6]. The result is sometimes expressed

in terms of a characteristic velocity* which is defined as

$$v_{ch} \equiv \sqrt{2\alpha t}$$

Because no consideration is made of flight time, the resulting “optimum” I_{sp} often results in a relatively long duration mission. It is possible to arrive at a different “optimum” I_{sp} which accounts for the cost of time by assuming a linear cost model for time (fixed \$/day). The results of these types of optimizations are generally not applicable to commercial C-EOR missions for one of several reasons.

First, some formulations assume that cost of power rises linearly with power level. This assumption is invalid for commercial EOR missions. Because a commercial payload is non-operational during orbit raising, there is substantial “free” power available for EOR that can be used without a cost or mass penalty. The incremental cost of power beyond this “free” level is very high because the extra power used for EOR is not useful for payload operation.

Second, some formulations do not properly consider the cost of flight time. On commercial missions, time has an inherent cost that must be considered in the optimization. Generally, the cost of time is a complicated function of a satellite operator’s business plan. For a new service business, the drive to be first-to-market may make the cost of time very high, but for a replacement satellite in an ongoing service, it may simply be modeled based on the time value of money. Whatever the cost that is assigned to time, the aim of the mission planner is to minimize this cost by maximizing the transportation rate (kg. gained/time) rather than the overall system dry mass. Maximizing transportation rate results in the highest benefit to the customer (mass) at the lowest cost (time and/or dollars).

Third, previous analytic formulations have ignored the contribution made by the chemical portion of the C-EOR mission. It will be shown that the optimum electric I_{sp} depends strongly on the chemical I_{sp} , and therefore can not be neglected when optimizing C-EOR missions.

* The term “characteristic velocity” has acquired more than one definition in the general literature. We define characteristic velocity as “the velocity gain of a rocket vehicle after complete consumption of its propellant, under absence of drag or gravity forces.” [2]

Considerable work has been done to evaluate the effectiveness of electric orbit raising with thrusters operating at a fixed I_{sp} . [8-11] Sophisticated codes exist which can optimize the spacecraft's steering profile to increase the mass delivered to orbit. [3, 4] Algorithms such as SEPSHOT and HYTOP calculate time-optimal transfers between orbits, constrained by Earth shadowing, oblateness, and solar cell degradation due to space radiation. The strategy employed by SEPSHOT and HYTOP, to minimize time-of-flight from an initial orbit to a target orbit, translates directly to propellant mass savings for missions with constant thrust and fixed I_{sp} . Both algorithms assume continuous thrust; hence, minimizing the time-of-flight minimizes the propellant expended. SEPSHOT also contains a limited chemical transfer module, which provides a chemical-electric transfer from an initial circular orbit. It applies a simple Hohmann-type transfer for the chemical portion, thus limiting the optimization of the chemical-electric transfer to a certain class of circular orbits.

Oleson has derived the optimum specific impulse for C-EOR missions to geostationary orbit (GEO) computationally and has considered the overall mission advantages associated with the use of constant power, variable I_{sp} EP devices for orbit raising and stationkeeping [5]. His work demonstrates the advantage of using one specific impulse level for orbit raising and a different level for stationkeeping, but does not consider the effects of continuously varying I_{sp} during orbit raising. To date, we are not aware of any published analytic derivations of the optimum I_{sp} for a C-EOR mission. In addition, previous work has assumed the use of a fixed I_{sp} thruster during orbit raising. This work provides an analytic derivation of the optimum I_{sp} and considers the benefits of varying I_{sp} during EOR.

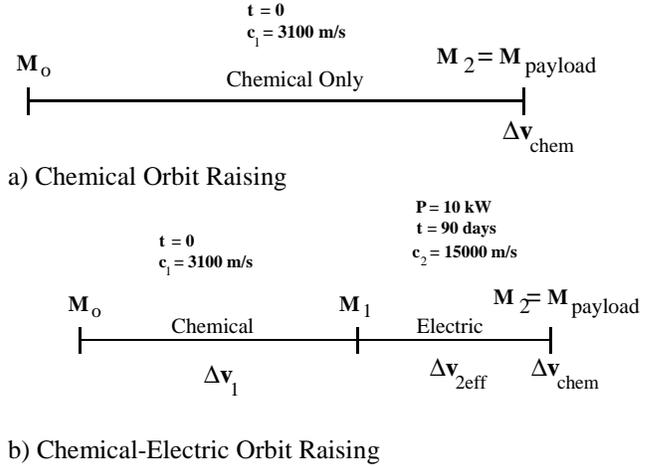


Figure 1: Simple Model of Typical Orbit Raising Missions

Analysis

Derivation of Rocket Equation for C-EOR

After a satellite has separated from its launch vehicle, a certain amount of chemical Δv is required to take the satellite from its separation orbit to GEO. The theoretical minimum Δv to GEO is a 3-D Hohmann transfer, and most chemical orbit raising missions closely match the theoretical profile. Figure 1 shows simplified representations of an all-chemical orbit raising mission and of a C-EOR mission. Figure 1a shows an all-chemical mission in which the spacecraft begins with an initial mass M_0 and ends with the payload mass M_2 . The payload mass is given by the rocket equation:

$$\frac{m_2}{m_0} = e^{-\Delta v_{chem}/c_1}$$

Where Δv_{chem} is the total chemical Δv required to reach orbit. EOR missions require more Δv than chemical missions because the very low thrust available does not allow for a Hohmann transfer to be performed, but saves mass by replacing relatively inefficient chemical Δv with much more efficient electric Δv . A C-EOR mission can therefore be divided into two parts, the total chemical Δv (Δv_1) which occurs first in the mission and the total electric Δv (Δv_2) which occurs second. This is shown in simplified form in Figure 1b. The spacecraft begins with an initial mass M_0 and ends the chemical phase of the mission with a mass M_1 . M_1 is also the starting mass at the beginning of the electric

phase of the mission. The spacecraft ends orbit raising with a final mass M_2 . The mass fraction for each phase of the mission is given by the rocket equation:

$$\frac{m_1}{m_0} = e^{-\Delta v_1/c_1} \quad (1)$$

$$\frac{m_2}{m_1} = e^{-\Delta v_2/c_2} \quad (2)$$

Equation (1) governs the chemical phase of the mission and equation (2) the electrical phase. Note that the sum of Δv_1 and Δv_2 is greater than Δv_{chem} . It is useful to define an effective Δv_2 which represents the amount of chemical Δv replaced by electric orbit raising.

$$\Delta v_{2eff} \equiv \Delta v_{chem} - \Delta v_1 \quad (3)$$

It is also useful to define an efficiency factor, η_v , which relates Δv_{2eff} to Δv_2 .

$$\eta_v \equiv \frac{\Delta v_{2eff}}{\Delta v_2} \quad (4)$$

η_v is a mission planning efficiency that represents the effectiveness of the thrust trajectory and steering profile. Its physical significance is discussed in section 5. Combining (2) and (4) gives the following form of the rocket equation for the EOR portion of the mission

$$\frac{m_2}{m_1} = e^{-\Delta v_{2eff}/\eta_v c_2} \quad (5)$$

In this expression, the planning efficiency represents a penalty applied to the electric thruster's exhaust velocity because the spacecraft is not flying the chemical mission profile.

Several more equations are required to relate power and time to the rocket equations. The thruster input power is related to the beam energy by

$$P = \frac{1/2 \dot{m}_2 c_2^2}{\eta_p} \quad (6)$$

Note that c_2 is an effective exhaust velocity, not the actual beam velocity. Time is related to mass flow rate by

$$m_2 = m_1 - \dot{m}_2 t \quad (7)$$

Using these expressions, it is possible to determine the thrust and specific impulse values which maximize the dry mass delivered to orbit on a given C-EOR mission. For the initial analysis, the following assumptions are made:

1. A fixed amount of power is available for EOR
2. A fixed amount of time is available for EOR
3. The EP device is always on (no coast periods)
4. The EP device's I_{sp} is constant throughout the mission
5. The thruster efficiency is not a function of I_{sp}
6. The planning efficiency is not a function of I_{sp}

The first two assumptions are design parameters determined by the design of the spacecraft and the contracted on-orbit delivery date. Fixing time ensures that the optimization will maximize the transportation rate. The next two assumptions are typical for minimum-time-to-orbit EOR missions. The fifth assumption is valid only over small ranges of variation, but is used to simplify the calculations. The validity of the last assumption is discussed below. Equations (1) (3) and (5) are combined to give the following expression

$$\Delta v_{chem} = c_1 \ln\left(\frac{m_0}{m_1}\right) + \eta_v c_2 \ln\left(\frac{m_1}{m_2}\right) \quad (8)$$

Equations (6) and (7) are combined to give

$$m_1 = m_2 + \frac{2\eta_p P t}{c_2^2} \quad (9)$$

Expressions (8) and (9) can then be combined to give the overall rocket equation for a C-EOR mission

$$e^{-\Delta v_{chem}/c_1} = \frac{m_2}{m_0} \left(\frac{\eta_p P t}{1/2 m_2 c_2^2} + 1 \right)^{\left(1 - \frac{\eta_v c_2}{c_1}\right)} \quad (10)$$

Expression (10) merits some discussion. As discussed previously, the optimum I_{sp} for an all-electric mission can be written in terms of a characteristic velocity which is a function of time, thruster input power, and

the spacecraft's specific power. Expression (10) can also be also written in terms of a characteristic velocity, but since the spacecraft's power level is fixed by assumption, specific power is replaced by the spacecraft power dedicated to orbit raising divided by the payload mass. Using this definition, the system's characteristic velocity is given by

$$v_{ch} \equiv \sqrt{\frac{2\eta_p Pt}{m_2}}$$

With this definition, expression (10) can be rewritten as

$$e^{-\Delta v_{chem}/c_1} = \frac{m_2}{m_o} \left(\frac{v_{ch}^2}{c_2^2} + 1 \right)^{\left(1 - \frac{\eta_v c_2}{c_1}\right)} \quad (10a)$$

Expression (10) is similar to the standard rocket equation, but with an additional term which captures the benefits of EOR. Minimizing this term increases payload mass delivered. Since the term $(1 - \eta_v c_2 / c_1)$ is less than one, EOR provides significant mass benefits only when the ratio of the system's characteristic velocity to the EP device's exhaust velocity is noticeably greater than zero (i.e. > 0.05).

Expression (10a) is general and applies not only to orbit raising, but to any mixed chemical-electric mission. All of the mission planning is summarized in two parameters: Δv_{chem} and η_v . In general, these parameters must be calculated computationally based on an assumed EOR flight and steering profile. As η_v increases, the payload mass fraction also increases. In the ideal case, $\eta_v = 1$, but this is a Hohmann transfer and is unachievable for low thrust missions. Finally, time, power, and thruster efficiency are all equivalent parameters in equation (10), so a 10% increase in EOR time has exactly the same effect as a 10% increase in EOR power. This is to be expected since with fixed I_{sp} , the total impulse delivered scales linearly with all three parameters.

Optimum Specific Impulse, Fixed Thrust

Since time is fixed by assumption, the specific impulse that maximizes the overall transportation rate is obtained from equation (10) by setting $dm_2/dc_2 = 0$. Two derivations of I_s^* are presented below. The first assumes that v_{ch} is small and is valid for short duration EOR missions. The second is a general expression valid whenever equation (10) applies.

When v_{ch}/c_2 is small, equation (10a) can be expanded using the binomial expansion. The result is:

$$e^{-\Delta v_{chem}/c_1} = \frac{m_2}{m_o} + \frac{v_{ch}^2}{c_2^2} - \frac{\eta_v v_{ch}^2}{c_1 c_2} \quad (12)$$

Equation (12) gives m_2 as an implicit function of c_2 . Equation (12) can therefore be differentiated implicitly using the following relationship: [7]

$$\frac{\partial m_2}{\partial c_2} = - \frac{\partial f / \partial c_2}{\partial f / \partial m_2} = 0 \quad (13)$$

And the result for I_s^* is

$$I_s^* = \frac{c_2}{g} = \frac{2c_1}{g\eta_v} \quad (14)$$

Expression (14) shows that I_s^* depends both on the I_{sp} of the chemical rocket and on the mission planning efficiency. This relationship can be understood by discussing the reasons for the existence of an optimum I_{sp} . Since time, power, and η_v are constrained by assumption, the magnitude of Δv_2 is determined by the electric I_{sp} . When c_2 is large, the EP device operates very efficiently, but delivers very little total impulse. Δv_2 is therefore small and the Δv delivered over the full C-EOR mission is dominated by the chemical thruster. This results in little overall mass savings. When c_2 is small, the EP device delivers a large total impulse, but operates relatively inefficiently. Although the Δv delivered over the C-EOR mission is dominated by the EP device, the propellant consumed by the EP device is high, so the overall mass savings is again relatively small. In the limit where $c_2 * \eta_v = c_1$, the effective I_{sp} of the electric thruster is the same as that of the chemical thruster, so EOR provides no overall mass benefit. The optimum I_{sp} for the EP device is at an intermediate level where the thruster delivers a relatively large total impulse but still operates with a relatively high fuel efficiency. The exact value of I_s^* depends on the characteristics of the chemical thruster because total impulse delivered by the EP device displaces fuel from the chemical thruster. The fuel efficiency of the EP device depends on the flight trajectory and steering profile and is therefore related to the planning efficiency, η_v .

Although spiral in-plane orbit raising can have an η_v greater than 0.9, this high a value for η_v appears to be

unrealistic for practical EOR missions. The optimization program SEPSHOT can be used to calculate thrust profiles for a wide variety of EOR missions that include changes in eccentricity and inclination [3]. SEPSHOT was used to calculate η_v for a variety of missions with durations between 5 and 85 days, electric specific impulses between 900 and 3000 seconds, and eccentricities between 0.036 and 0.169. All cases used a starting inclination of 0.39 degrees. These values are typical starting points for EOR missions to GEO. The results are summarized in Figure 2. The range of calculated η_v 's is relatively narrow, ranging from 0.45 to 0.55 for all the cases analyzed. The relatively narrow range justifies the initial assumption that η_v varies little with changes in I_{sp} . A typical value for η_v appears to be 0.5. Since chemical thrusters used for orbit raising typically have an I_{sp} of about 310 seconds, equation (14) gives an optimum I_{sp} of approximately 1250 seconds for the EP device.

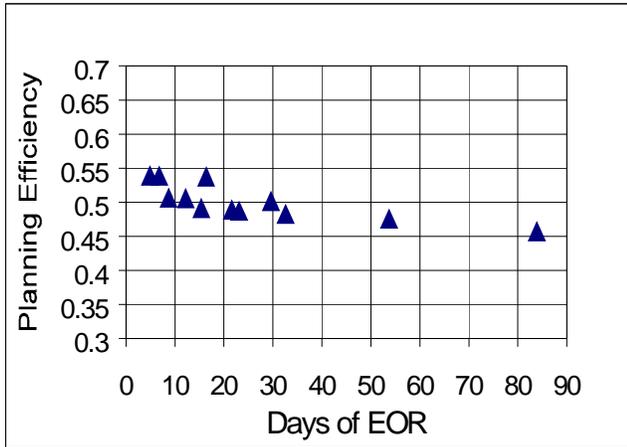


Figure 2: SEPSHOT Calculated Mission Planning Efficiencies

Power and time do not appear in (14) because they were assumed to be small in (12). However, (13) can also be used with equation (10) to derive the general expression for the optimum I_{sp} for a C-EOR mission. The result is

$$2v_{ch}^2 \left(1 - \frac{\eta_v c_2}{c_1} \right) + \frac{\eta_v c_2}{c_1} (v_{ch}^2 + c_2^2) \ln \left(\frac{v_{ch}^2}{c_2^2} + 1 \right) = 0 \quad (15)$$

Equation (15) gives an implicit relationship for power, time, and I_{sp} . Since time and power are constrained by assumption, for a given η_v , the I_{sp} can be determined numerically using a spreadsheet. Figure 3 shows the

optimum I_{sp} as a function of time and η_v for a hypothetical EOR mission to GEO. The following parameters are assumed for the calculation: $M_2 = 4000$ kg. Power = 10 kW, $\eta_p = 0.5$, chemical $I_{sp} = 310$ sec. Customer requirements limit commercial missions to a relatively short duration (< 90 days).

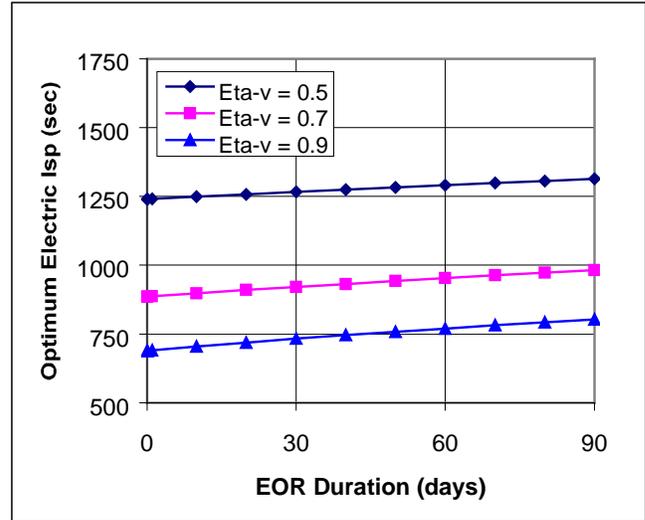


Figure 3: Optimum Specific Impulse vs. EOR Duration

Although figure 3 is plotted against time, as discussed earlier, time and power are equivalent parameters in this analysis. For this class of missions, v_{ch} is relatively small and the optimum specific impulse varies relatively weakly with time and power. Figure 3 shows that expression (14) is valid over most of the range of interest for GEO missions. Figure 3 also shows that the optimum specific impulse varies strongly with the mission planning efficiency, but confirms the basic result that for the typical GEO missions shown in Figure 2, I_{sp}^* is approximately 1250 sec.

Most EP devices operate efficiently over a limited specific impulse range. On a given mission, it may not be possible to operate at exactly the optimum I_{sp} , so it is desirable to determine the sensitivity of the payload mass to variations in I_{sp} . To examine this effect, equation (10) was used to determine the dry mass delivered as a function of I_{sp} on a hypothetical C-EOR mission. The results of this analysis are shown in Figure 4. The following parameters are assumed for the calculation: $M_0 = 6000$ kg., Power = 10 kW, EOR duration = 90 days, chemical $I_{sp} = 310$ sec, $\Delta V_{chem} = 1800$ m/s, $\eta_v = 0.5$ and $\eta_p = 0.5$.

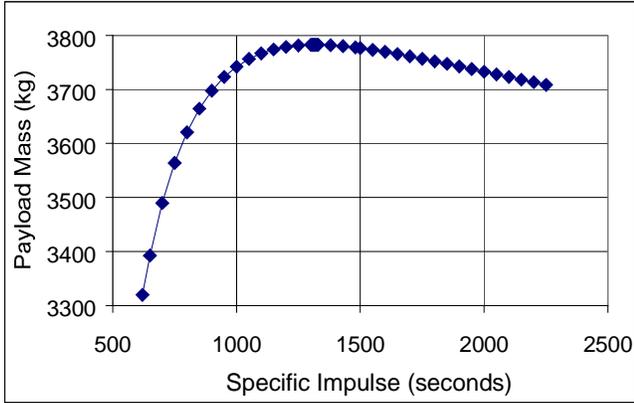


Figure 4: Dry Mass Delivered vs. Specific Impulse for Typical 90 Day EOR Mission

For this particular mission, equation (14) yields an approximate optimum I_{sp} of 1240 sec and equation (15) gives an exact optimum of 1320 sec. A mission flown at the optimum I_{sp} delivers a payload of 3780 kg, which corresponds to a transportation rate of 5.1 kg./day, or a total mass benefit of 460 kg. for this 90 day mission. The slope of the performance curve is quite steep below the optimum point but relatively flat around and above the optimum, so an optimized EP device would operate at or slightly above the theoretical optimum. It should be noted that this result assumes that the efficiency of the thruster varies little with changes in I_{sp} . In reality, the efficiency of the thruster tends to improve with I_{sp} , so the actual optimum will be somewhat higher than the value calculated here.

If the mission is flown using a typical 300 V Hall Thruster with $I_{sp} = 1500$ sec, the payload mass is 3775 kg., or virtually identical to the optimum case. If the mission is flown with a typical Ion Thruster with $I_{sp} = 2000$ sec, the payload mass is 3730 kg., which corresponds to a transportation rate of 4.6 kg./day and is 10% lower than the optimum case. An arcjet operating at 600 sec. I_{sp} would deliver no mass benefit over chemical orbit raising. For the class of commercial EOR missions shown in Figure 2, Hall Thrusters appear to be the current technology which comes closest to matching the optimum I_{sp} .

Optimum Specific Impulse, Variable Thrust

The analysis presented in the previous section assumes that the I_{sp} of the EP device is fixed throughout the mission. However, variable I_{sp} thrusters are currently under development [12], and previous work has examined the benefits of operating at one specific impulse level for orbit raising and a different level for

on-orbit operations [5]. This section will examine the benefits of using a constant power, variable I_{sp} device during orbit raising operations.

Expressions (14) and (15) demonstrate that the optimum specific impulse for a given orbit raising mission is strongly dependent on the mission planning efficiency, η_v . In the previous section, a single overall η_v was calculated for each mission. Physically, η_v is a measure of the effectiveness of the firing profile chosen for EOR. In general, it is more efficient to deliver Δv in a series of impulsive, high thrust maneuvers than it is to deliver Δv in a continuous, low thrust firing. Inclination changes, for instance, are accomplished most efficiently by thrusting perpendicular to the orbital plane at an orbit's ascending and descending nodes. Thrust delivered near the nodes is more effective than thrust delivered away from these points.

If a low thrust device is used for an inclination change mission, the effectiveness of the thrust delivered varies over time. Since η_v is a measure of the effectiveness of the thrust profile, it follows that η_v also varies over time and can be defined as a local parameter. Equations (14) and (15) relate the optimum specific impulse to variations in η_v , producing a time varying optimum I_{sp} profile. For a power and time constrained mission, varying the thruster's specific impulse to follow the optimum throttling profile increases the mass delivered to orbit. This result is demonstrated below using a simplified EOR mission.

Consider an orbit raising mission in which a satellite begins in a 24 hour, slightly inclined circular orbit and ends in GEO, as shown in Figure 5. The time and power available for EOR are fixed, but the starting point for EOR is allowed to vary depending on the specific impulse chosen for EOR. For simplicity, it is assumed that the EOR duration is one day, so the inclination change is small and equation (14) is valid.

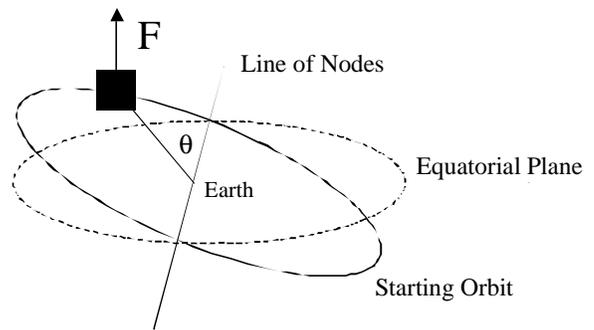


Figure 5: Inclination Change Only Orbit Raising

Mission

η_v is the ratio of the Δv for an optimized chemical mission divided by the Δv for the equivalent electric mission. The optimum chemical burn profile is a single burn on the line of nodes with the thrust vector oriented perpendicular to the orbital plane. For small changes in inclination, the chemical Δv is given by.

$$\Delta v_{2eff} = \sqrt{\frac{\mu}{a}} \Delta i \quad (16)$$

The electric Δv is a function of the chosen thrust profile. If the electric thruster is assumed always on, the optimum EOR profile is one in which the thrust vector is oriented perpendicular to the orbit and points towards the equatorial plane. The thrust direction is reversed as the satellite passes through the ascending and descending nodes. With this low thrust mission profile, the rate of change in inclination as a function of orbital position is given by

$$\frac{di}{dt} = \frac{F}{m} \sqrt{\frac{a}{\mu}} \cos \theta \quad (17)$$

where θ is the angle from the line of nodes as defined in Figure 5. The total change of inclination over one day is obtained by integrating (17) over one orbit. The Δv required to obtain a given change in inclination is obtained by inverting the result and is given by

$$\Delta v_2 = \frac{\pi}{2} \sqrt{\frac{\mu}{a}} \Delta i \quad (18)$$

The planning efficiency is the ratio of (16) to (18), or

$$\eta_v = \frac{2}{\pi} = 0.637 \quad (19)$$

Since the EOR duration is short, the I_s^* is given by expression (14). Assuming a typical chemical I_{sp} of 310 sec, the optimum I_{sp} is:

$$I_s^* = \frac{2c_1}{g\eta_v} = 973 \text{ sec} \quad (20)$$

Equation (19) is based on an average η_v obtained by integrating expression (17) over time, so (20) represents the optimum specific impulse for thrusters operating at a fixed I_{sp} . A local value for η_v can be

obtained directly from (17) by applying Newton's third law and integrating over a short time to give

$$\Delta i = \Delta v \sqrt{\frac{a}{\mu}} \cos \theta \quad (21)$$

Solving for Δv and then taking the ratio of (16) to (21) gives the local value of η_v

$$\eta_{vlocal} = \cos \theta$$

This in turn leads to a position dependent expression for the optimum specific impulse

$$I_s^* = \frac{2c_1}{g \cos \theta} \quad (22)$$

Expression (22) shows that the optimum specific impulse varies continuously with time. When the spacecraft is near the nodes (θ near 0), the thrust efficiency is high, and a low I_{sp} maximizes the total impulse delivered at these point. When the spacecraft is far away from the nodes (θ near 90 degrees), a high I_{sp} compensates for the inherent inefficiency associated with thrusting away from the optimum locations. Note that when the spacecraft is 90 degrees away from the nodes, thrusting has no effect on the orbit's inclination, so I_s^* is infinity.

The benefit of using the optimum I_{sp} profile is demonstrated by using equation (17) to directly simulate a one-day inclination-change-only EOR mission to GEO. At a fixed power level, the thrust level for a given I_{sp} is calculated using equation (6)

$$F = \frac{2P\eta_p}{c_2}$$

In this case, c_2 varies as a function of time. Using this expression, a spreadsheet is used to numerically integrate equation (17) around a 24 hour orbit. An angular step of $d\theta = 0.025$ radians is used and because of the short mission duration, the spacecraft mass is held constant. A different specific impulse level and/or profile is assumed in each run and Δi and the propellant consumed are calculated. Δi is used with equation (16) to calculate an equivalent Δv_{2eff} and the result is used to calculate the amount of chemical propellant required to obtain the same change in inclination. The net mass savings is given by the difference between the amount of chemical propellant

that would have been required for the mission and the amount of electric propellant which was actually used in the simulation.

Note that the amount of Δi is slightly different in each of the runs. As was stated earlier, the starting point for EOR is allowed to vary for each specific impulse profile. If additional Δi is required for the satellite to reach GEO, it is assumed that it is accomplished using chemical propulsion. This assumption is strictly valid only in cases where the total inclination change for both the chemical and electrical portions of the mission are small. This is equivalent to assuming that Δv_{chem} is relatively small.

A series of simulations were run in which it was assumed that the thruster's specific impulse was fixed. Figure 6 summarizes the results for a range of different I_{sp} 's. The following parameters are assumed for this calculation: $M_2 = 3500$ kg., Power = 9 kW, EOR duration = 1 day, chemical $I_{sp} = 310$ sec, $a=42164$ km (GEO orbit) and $\eta_p = 0.5$.

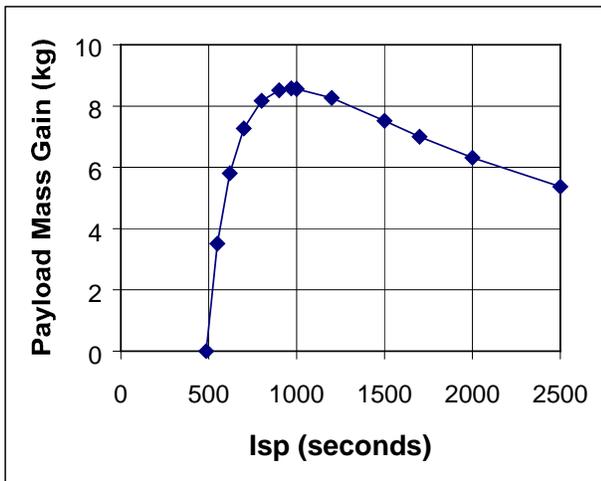


Figure 6: Payload Mass Gain vs. I_{sp}
Small Inclination Change to GEO,
 Duration = 1 day

Since time is fixed by assumption, payload mass is equivalent to transportation rate. As expected from (20), the highest transportation rate is achieved when the specific impulse is 975 seconds. Again, the slope is relatively steep below the optimum point and relatively shallow above it. The total mass gain is small because the EOR duration is limited to one day. The transportation rate is higher than that shown in Figure 4 because of the short EOR duration and simplified mission assumptions.

Additional simulations demonstrate the benefits of varying I_{sp} during orbit raising. Figure 7 shows a specific impulse profile which follows the ideal profile given by equation (22). The simulation begins on the line of nodes at time = 0. This profile is unrealistic since the required specific impulse at $t = 6$ hrs. and $t = 18$ hrs. is infinity. Figure 8 shows a specific impulse profile which follows equation (20) but is constrained to a maximum specific impulse of 2000 seconds. The I_{sp} of the chemical thruster is assumed to be 310 sec. When $(2000 \text{ sec}) \times (\eta_v)$ is less than the specific impulse of the chemical thruster, firing the thruster provides no mass benefit, so the thruster is turned off. This results in coast periods of approximately three hours. Figure 9 shows a similar profile in which the thruster is also constrained to a minimum specific impulse of 1000 seconds. This profile could be achieved using a single stage Hall thruster. Table 1 summarizes the payload mass gain achieved using each of these profiles and compares them to the optimum fixed I_{sp} case shown in Figure 6 above. Table 1 was calculated using the same assumptions used for Figure 6.

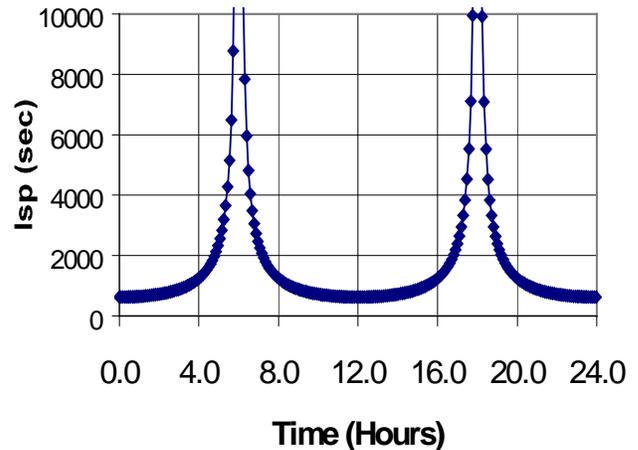


Figure 7: Ideal Thruster Throttling Profile

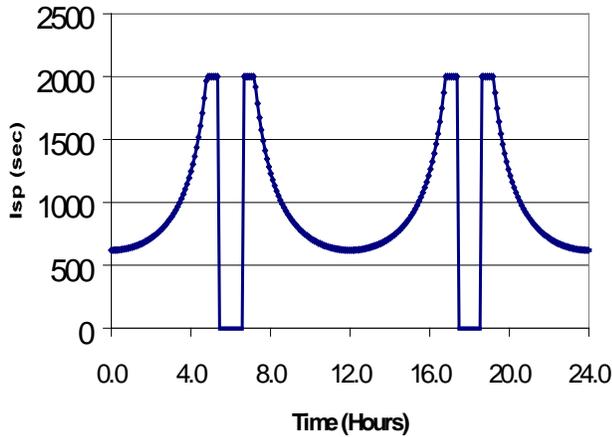


Figure 8: Thruster Throttling Profile
Max I_{sp} = 2000 sec

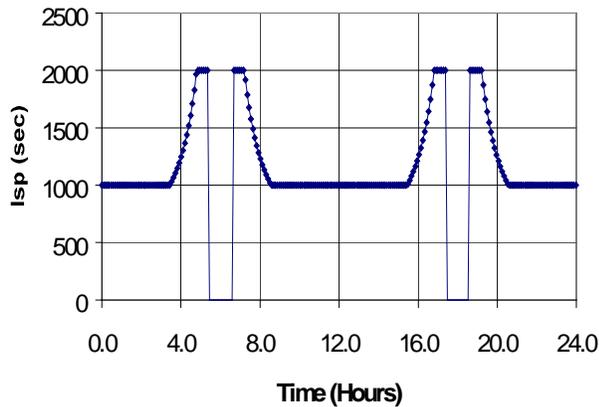


Figure 9: Thruster Throttling Profile
Max I_{sp} = 2000 sec, Min I_{sp} = 1000 sec

Table 1 shows that using a variable I_{sp} profile increases the transportation rate by up to 23% over the unthrottled case with no change to the time or power required for orbit raising. This savings is achieved by using high thrust when the local η_v is high and using high I_{sp} when η_v is low. Limiting the maximum allowable I_{sp} has relatively little effect on the total mass savings because most of the Δi occurs when the thruster is operating at high thrust level. Limiting the minimum I_{sp} substantially lowers the transportation rate for the same reason. The results indicate that for this particular mission, a throttled Hall thruster could provide a mass savings of 12% without impacting the satellite's power or the mission duration.

Table 1: Transportation Rate for Fixed and Variable Specific Impulse Missions

I_{sp} Profile	Transportation Rate (kg/day)	Improvement over Fixed I_{sp} case (%)
Fixed, 975 sec	8.57	0%
Variable, Infinite Range	10.57	23.3%
Variable, < 2000 sec	10.53	22.9%
Variable, 1000-2000 sec	9.61	12.1%

Although the results shown in Table 1 are based on a specific and simplified EOR mission, the important result is the observation that throttling specific impulse can significantly increase dry mass delivered to orbit when a fixed amount of time and power are available for EOR. This observation should be valid for many EOR missions. Any orbit raising mission which includes changes in inclination or eccentricity is likely to exhibit local variations in η_v . Since the optimum specific impulse is directly related to η_v , these missions will all have an optimum throttling profile which delivers better performance than a fixed throttle level. It is theorized that realistic EOR missions of the type shown in Figure 2 may experience mass benefits similar to those shown in Table 1 when I_{sp} throttling is included in the mission profile. For a 90 day EOR mission of the type shown in Figure 4, this could lead to a mass benefit of approximately 100 kg. Further work is needed to confirm this theory and to determine how one would calculate the optimum I_{sp} profile for a realistic EOR scenario. Once a general method for calculating the optimum profile has been developed, it will be possible to fully quantify the benefits of variable I_{sp} profiles on EOR missions.

Conclusions and Future Work

An analytic derivation has been made of the optimum specific impulse (I_{sp}) for a time and power constrained mixed chemical-electric orbit raising (C-EOR) mission. The results show that the optimum I_{sp} for a C-EOR mission depends primarily on the I_{sp} of the chemical thruster and on a mission planning efficiency, η_v , which is determined by orbital dynamics and by the thrust steering profile. Based on the analysis presented in this paper, the following conclusions have been reached:

- The optimum I_{sp} for most commercial EOR missions to GEO can be estimated using the following simple expression:

$$I_s^* \approx \frac{2c_1}{g\eta_v} \quad (14)$$

Where η_v is the mission planning efficiency. Equation (14) is a simplified expression valid for short duration and/or low power missions. The exact expression for the optimum I_{sp} is given implicitly by equation (15).

- Hall Thrusters provide close to optimum performance for commercial EOR missions to GEO flown with a fixed I_{sp} .

For realistic EOR missions, η_v is typically between 0.45 and 0.55. The corresponding optimum I_{sp} ranges from 1120-1380 sec. Although a typical 300 V Hall Thruster operates at a slightly higher specific impulse, the payload mass delivered is close to the optimum value. This result assumes that the thruster's efficiency is constant with changes in I_{sp} .

- Use of a constant power, variable I_{sp} device during orbit raising provides better mass performance than a fixed I_{sp} device.

In general, it is possible to define η_v as a local parameter that varies with time and orbital position. Because of the strong dependence between η_v and the optimum I_{sp} , it follows that an optimum specific impulse throttling profile exists for most EOR missions. Use of this throttle profile allows delivery of a higher payload mass to orbit with no time or power penalty.

- In one simplified mission scenario, using a variable I_{sp} profile increases the effectiveness of EOR by over 20%.

This result is demonstrated by simulating a simple inclination change only EOR mission to GEO. In this scenario, a mass gain of 23% is achieved by following the optimum firing profile. A firing profile limited to throttle levels between 1000 and 2000 seconds I_{sp} demonstrates a performance gain of 12%. These results are only strictly applicable to this particular mission scenario.

Based on our understanding of the physical basis for the optimum throttle profile, we theorize that the following statements may also be true.

- The use of constant power, variable specific impulse devices may allow mass improvements of 20% or more on commercial EOR missions
- The ideal thruster for throttled EOR missions to GEO may need to operate efficiently between 600 and 2000 sec I_{sp} .

If a performance improvement of 20% can be achieved for the types of EOR missions considered in this paper, the net mass gain with 90 days of EOR would be approximately 100 kg. Considerable work is needed to confirm these findings and apply this model to a realistic mission profile. Work is in progress to improve the existing model by incorporating coast periods and obtaining a better understanding of how mission planning efficiency varies with mission duration and starting location. It is also necessary to incorporate variations in thruster efficiency into the model.

It is clearly desirable to develop a general method for calculating the optimum throttle profile for different types of EOR missions. Although it is theoretically possible to derive a mission's I_{sp} throttling profile and thrust steering profile as part of the same optimization, we are not aware of any previous work in this area. An additional challenge is to develop thrusters capable of operating over the wide specific impulse range necessary to support throttling during EOR. It appears desirable to operate efficiently at I_{sp} 's as low as 600 sec. and as high as 2000 sec. for orbit raising, and possibly higher for on-orbit operations. The community should consider developing both the mission analysis tools and the electric propulsion hardware needed to support throttled EOR missions.

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