An Electric Propulsion System Based on Controlled Fusion and Electromechanical Energy Conversion

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Abstract: E**xamination continues in detail for the use of controlled fusion for high-energy missions to the outer planets in the context of fast trip times resulting from significant, continual acceleration (> 0.01g$_o$). Transit times to Jupiter measured in a few months appear possible, but require the use of advanced fuels, e.g., D-He^3, to reduce the excessive burden of radiator mass needed to process the heat of fusion neutrons. Such advanced reactions, however, need plasma temperatures in excess of 100 keV, so adiabatic compression is invoked to match the desired final state to much more modest initial temperature values. This compression would be achieved by stabilized implosion of liquid metal liners, as demonstrated decades ago in the Linus program at the Naval Research Laboratory. Distinctions are drawn between the present propulsion design options and the earlier Linus fusion power reactor based on D-T fuel for which charged particles were a relatively small component of the fusion energy gain and the output power (mostly from neutrons) was extracted as heat. Applications of the present system concept to interstellar missions and to power production are also considered.**

Nomenclature

B	= magnetic field
E_n	= charged-particle energy from nuclear reactions
F	= thrust
h	= effective length of outer coil representing liner or combustor
H_R	= heat to be rejected
K	= correction factor for inductance of finite-length coil
$l_{i,\,f}$	= plasma length, respectively, initial in combustor, final in combustor
L	= self-inductance
M_{pL}	= mutual inductance of plasma and outer coil
$M_{E,\,o,\,pay,\,prop,\,R,\,t}$	= mass, respectively, of engine, original total, payload, propellant, radiator, tankage
N_D	= total number of deuterium ions in FRC
P_J	= jet power
q	= heat deposited in FRC by nuclear reactions
$r_{D,\,L,\,o,\,p}$	= radius, respectively, of (outer edge) of drive-piston, outer coil (liner or combustor), initial liner surface, plasma (major)
$T_{i,\,f}$	= FRC temperature, respectively, initial in combustor, final in combustor
$V_{i,\,f}$	= plasma volume, respectively, initial in combustor, final in combustor
w_R	= resistive energy loss in combustor skin-layer relative to W_L
$W_{L,\,p,\,T}$	= energy, respectively, in self-inductance of outer coil, in plasma, and total (plasma and magnetic)
α_R	= specific power of thermal radiator system
β_L	= FRC plasma energy parameter, $(\gamma - 1)W_p/2W_L$
δ_L	= resistive diffusion skin-depth in combustor surface

I. Introduction

As electric propulsion, in the form of ion and Hall thrusters, continues to find acceptance for near term, robotic missions, it is appropriate to look further out for concepts that would enable crewed exploration of the Solar System. In a previous paper[1], we discussed the concerns for crewed missions to the outer planets, including long term effects of radiation and weightlessness and the increased risk of high energy radiation events from solar flares. These concerns led to the specification of trip times of less than a few months and the associated continual acceleration at >1% of earth surface gravity g_o. For a mission to the Jovian system, e.g., Europa, the Δv estimated for the outbound and the return trip is about 500 km/s, suggesting comparable values for optimum exhaust speeds and requiring (jet) specific powers exceeding 20 kw/kg. Such specific power levels resemble optimistic possibilities for just the space-radiator systems and thus place an emphasis on minimizing heat production by the propulsion and power system. The approach introduced in the earlier paper[1] involves the generation of power by so-called "advanced fuel" fusion reactions (e.g., D-He^3) to minimize neutron production and to allow the nuclear energy to deposit in the propellant as the initial portion of an open thermodynamic cycle, equivalent to chemical rocket engine operation.

In the present paper, we examine in greater detail the several mechanisms needed to achieve conceptually the desired fusion propulsion system. The primary mechanism is the use of stabilized, repetitive implosion of liquid metal cylinders, known as liners[2], with radial compression ratios of 10:1, to heat so-called *compact toroid*[3] plasma/magnetic field targets by adiabatic compression. The compact toroid arrangement considered here is an elongated field-reversed configuration (FRC) that has been studied for several decades[4,5]. Adiabatic compression provides plasma at temperatures over 50 keV, contained by magnetic fields at megagauss levels obtained by the associated compression of magnetic flux. The subsequent nuclear energy deposition raises the plasma temperature to 180 keV and also performs work on the confining magnetic field, thereby providing the increased energy for a pulse of directed kinetic energy obtained by expansion of the compact toroid in a magnetic nozzle.

II. Background

The technical elements of the present conceptual design draw on rather esoteric experiences, so it is useful to recapitulate some of the past successes justifying pursuit of the present approach. In particular, the operation of stabilized, repetitive implosion of liquid liners was demonstrated at the Naval Research Laboratory (NRL) in the late 1970's[2], and elongated FRCs, displaying exceptional stability, lifetime and robustness, have been created in many laboratories around the world[4,5]

Stabilized liner implosions

In order to compress a plasma/magnetic field target efficiently using an imploding liner of much higher density, instabilities must be overcome. Two types of instability have been observed experimentally. For implosion of liners that stay in the solid state, elastic-plastic modes arise due to the reduction of the liner circumference. These stabilize for any azimuthal mode number when the liner thickness exceeds a quarter-wavelength. Sufficiently rapid reduction of the circumference and increase of the liner thickness can successfully overcome at least the higher frequency modes. Radial compressions of 30:1 have been achieved[2]. After peak compression, however, the liner material rebounds as shrapnel. This is not felicitous for laboratory fusion experiments, even launched electromagnetically, that would require kinetic energies equivalent to many tens of pounds of high explosive.

The use of liquid metal liners can suffer from Rayleigh-Taylor instability as the inner surface of the liner decelerates in compressing the lower density target, and also during the initial acceleration and subsequent deceleration of the outer surface of the liner. The former circumstance is avoided by rotating the liner material so that the inner surface experiences a centripetal acceleration that reverses the adverse direction of the effective gravity near peak compression. This technique does not apply to the outer surface. Instead, instability there is eliminated by driving the liner with a free-piston in continual contact with the liquid liner. Such an arrangement for stabilizing a liquid metal liner implosion is shown schematically in Fig. 1 for a system called Helius[2] at NRL, c. 1979. This system uses an annular free-piston, driven axially by high pressure helium, to displace the inner surface of a rotating liquid inward onto a volume of trapped gas and/or magnetic flux. It successfully compressed magnetic flux using a liner of NaK at its eutectic mixture. Figure 2 displays the repetitive implosion and re-expansion of a stabilized liner of water, 10 cm long, compressing air in a separate system[2]. The cyclic motion persisted through thirteen oscillations on a single gas charge. Linus-0[2], a much larger version of this "water model" was also successfully operated at NRL before the program closed for lack of an appropriate plasma target.

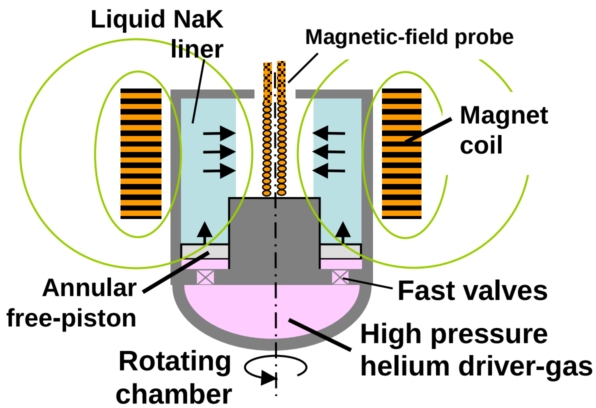

Figure 1. Schematic of the Helius apparatus at NRL for stabilized implosion of NaK liners onto trapped gas and magnetic flux[2].

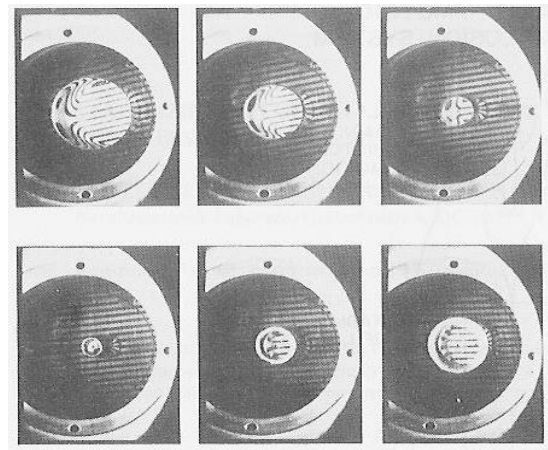

Figure 2. High speed photographs of repetitive implosion and re-expansion of stabilized liquid liner (H_2O) compressing air[2]. The depth of the cylinder, viewed at an angle here, is 10 cm.

Field-Reversed Configuration

The Linus program at NRL originally was inspired by work at IAE Kurchatov in Moscow on liners to compress an open-ended theta pinch, the so-called "theta-pinch-with-liner"[6]. Later, however, interest shifted to a closed-field arrangement termed a *compact toroid* by its developers[3]. This was a form of field-reversed configuration (FRC) first noticed in early theta-pinch experiments[4]. A basic arrangement is depicted in Fig. 3. Since the late 1970's, there have been a large number of experiments[4,5] comprising various techniques for FRC formation and heating and similarly large number of attempts to model FRC behavior[5]. Many processes contributing to FRC lifetime indicate a continuing need for research, including stability in the face of field-line curvature that is adverse from the standpoint of MHD theory, coupling of the closed-field and open-field line regions, and transport that is somewhere between classical and Bohm-like. Issues include control of plasma rotation, the importance of finite Larmor-radius effects, and possibly critical profiles of field and density. It appears that there is some advantage in operating with elongated FRCs for which the length-to-diameter ratios are much greater than unity, so the present paper will assume that the formation and operation of such arrangements will be successfully available by the time decisions on advanced space propulsion must be taken.

III. Liner-Driven Electric Thruster

If an elongated FRC of adequate lifetime is possible, then the stabilized liner implosion technique can be used to compress the plasma adiabatically to high temperatures. Release of the FRC to a magnetic nozzle before the liner surface has re-expanded allows conversion of the plasma and magnetic energies gained by the FRC into the directed energy of a pulsed rocket exhaust, as suggested in Fig. 4. Mechanical energy from a thermal power plant can thus be transformed in a high speed flow. Simple analysis of an FRC indicates that during radial compression, the FRC length will decrease as the radius to the 2/5ths-power. A radial compression ratio of 30:1 would thereby raise the plasma temperature by a factor of $(30)^{1.6} = 231$; the surrounding magnetic field increases due to flux compression in accord with the balance of plasma and magnetic pressures by a factor of 900. An FRC with an initial temperature of 100 eV, for example, would attain a peak temperature of 23 keV. If lithium is the propellant (an eminently storable choice), full expansion of the FRC, with its energy split equally between magnetic energy and plasma energy (Li plus three electrons) offers an exhaust speed of 2800 km/s. While this speed would require either very high values of specific power or long thrust times to justify its use for an optimized system[7], it nevertheless interesting that such speeds might be possible from an electromechanical process. This process depends on stable implosion of a liquid metal and containment of the FRC plasma.

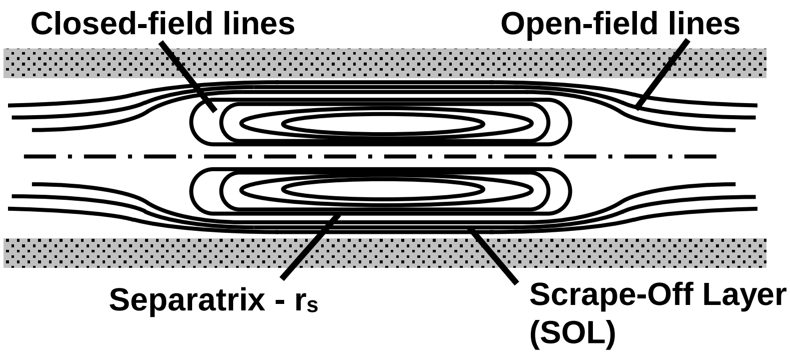

Figure 3. Depiction of elongated field-reversed configuration (FRC) with closed-field lines retaining plasma and open-field lines allowing plasma escape.

Note also that the efficient use of a magnetic nozzle requires that the plasma is borne *by* the nozzle field lines and not born *on* these field lines. Otherwise significant energy is lost to detaching the plasma from the field by resistive diffusion.

The work required to energize the FRC by compression is provided by the expansion of gas in the piston drive and comes from the thermodynamic cycle of the power plant. (An alternative approach would drive the piston electromagnetically, which might involve substantial capacitive energy storage, at specific energies much less than available by pneumatic or inductive techniques[1]). If the FRC was able to derive energy from fusion reactions, then the liner-driven electric thruster could operate much like a conventional rocket, with an open thermodynamic cycle and nuclear energy replacing chemical energy.

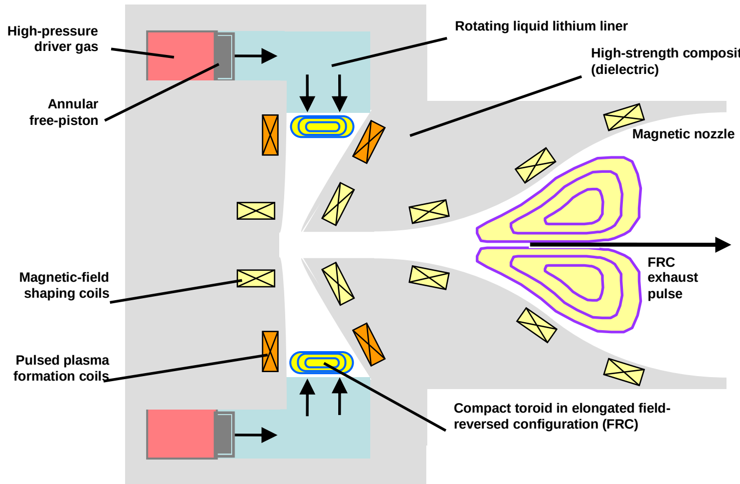

Figure 4. Concept of a liner-driven electric thruster in which a stabilized liner implosion system compresses an FRC adiabatically to high specific energy for a very high speed exhaust pulse.

IV. Liner-Driven Fusion Propulsion

The Linus fusion reactor design[2,8] was based on D-T reactions because these are the most accessible, highly energetic processes, providing a 14.1 MeV neutron and a 3.5 MeV α-particle (He^4 nucleus). In this design, the imploding liquid liner not only supplies the basis for adiabatic compression to megagauss field levels, but it serves as the reactor blanket and first-wall, thereby eliminating major issues associated with solid-density structures exposed to intense neutron flux. For the stabilized liner approach, however, the liner material must extend from the piston-face to the inner surface of the liner, so high radial compression ratios imply relative thick liners. The compressibility of the liquid metal then introduces an inefficiency of delivering the full kinetic energy of the liner to the target. In particular, the inner surface can slow and turn-around before pressure waves can reach the outer portions of the liner. This process has been analyzed in detail for rotationally-stabilized liners[9] and subsequent design studies suggest that liner implosion systems will optimize at much lower values of peak magnetic field than originally contemplated (e.g., 0.5-1 MG vs 3-10 MG). The resulting conceptual design of a D-T fusion reactor would then offer about 250 Mw of electrical output (at a circulating power fraction of 10%). While much lower than "conventional" fusion reactor designs (e.g., tokamaks), this is still rather unattractive for entry-level power plant technology, and must compete against other terrestrial power concepts.

For advanced space-propulsion missions, such as a crewed voyage to the outer planets, fusion may represent the only approach (within our present physical understanding). The issue then becomes one of heat rejection, with the substantial fraction (>80%) of the output from D-T reactions as neutrons representing too high a burden on the specific power of the propulsion system[1]. Instead, the use of D-He^3 can be explored for which the primary reaction products are protons (14.7 MeV) and He^4 (3.6 MeV). There will be some neutron production due to D-D reactions, half of which provide a neutron (2.45 MeV) and the other half a tritium nucleus that can then react (eventually) with deuterium resulting in the previously mentioned 14.1 MeV neutron. Even with these other reactions, however, the energy from neutron amounts to less than a few per cent of the energy from the D-He^3 reaction. (The exact percentage depends on the necessary burn-time compared to the time for growth and reaction of tritium.)

Another aspect in which the present approach with D-He^3 differs from the earlier D-T design is the substantial deposition of the reaction energy into the FRC. For a D-T system, the α-particles can replace the energy lost from the plasma by transport, while the neutrons supply fusion energy without affecting the magnetic field configuration. With D-He^3, however, the fusion energy stays within the FRC, raising the plasma temperature and the associated plasma pressure relative to the magnetic pressure (i.e., the plasma beta). At some point, the magnetic configuration cannot tolerate the fusion gain. Thus, an FRC arrangement is needed that can adjust as the plasma heats. Also, it is useful to limit the excursion in temperature in order to remain near the optimum value of reaction-rate divided by temperature ($<\sigma v>/T$) for a constant number of particles. The desired temperature range is approximately 50 – 180 keV. The initial value would be obtained by adiabatic compression of the FRC, while the upper value corresponds to a limit on nuclear gain within a single pulse.

The necessary nuclear gain must be compared to the energetic costs of operating the system. A feature of the original Linus reactor design[2,8]was the replacement of mechanical and resistive losses by the extra work on the rebounding liner available from the payload thanks to α-particle deposition. (This deposition was invoked at a time when finite Larmor-radius requirements in the FRC were not quantified, a situation that may not yet be understood.) With the large scale-up of the NRL experiments to reactor levels, the principal concern would be resistive diffusion during the time required for adequate fusion burn. The relative loss associated with such diffusion is mitigated by radial dimensions large compared to diffusive depths, thereby setting a size for the FRC and liner system. The diffusive depth will increase with the square-root of the burn-time, suggesting that operation at the highest value of pressure and magnetic field is beneficial.

The studies of compressible liners[9] indicate that the increased dwell times at high peak pressures do not totally compensate for the reduced efficiency of utilizing the liner kinetic energy. Previously, this forced the optimum design to peak fields below a megagauss. Since the time of such design, however, advances in materials and mechanical design have resulted in repetitively-operated, static coils with relative long pulses (few msec) at fields above a megagauss[10]. It may therefore be possible to separate the burn-time from the liner dynamic time by inserting the actual burning plasma into a "combustor" based on a high-strength static coil.

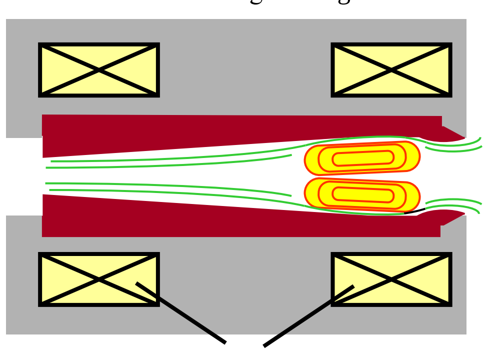

Figure 5. High-strength static coil "combustor" section

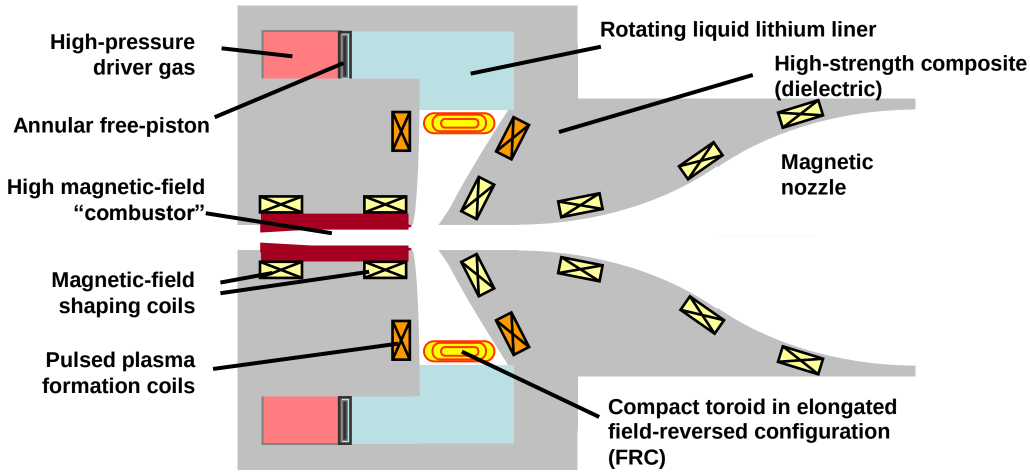

Figure 6. Conceptual design of a fusion rocket based on the liner-driven electric thruster of Fig. 4.

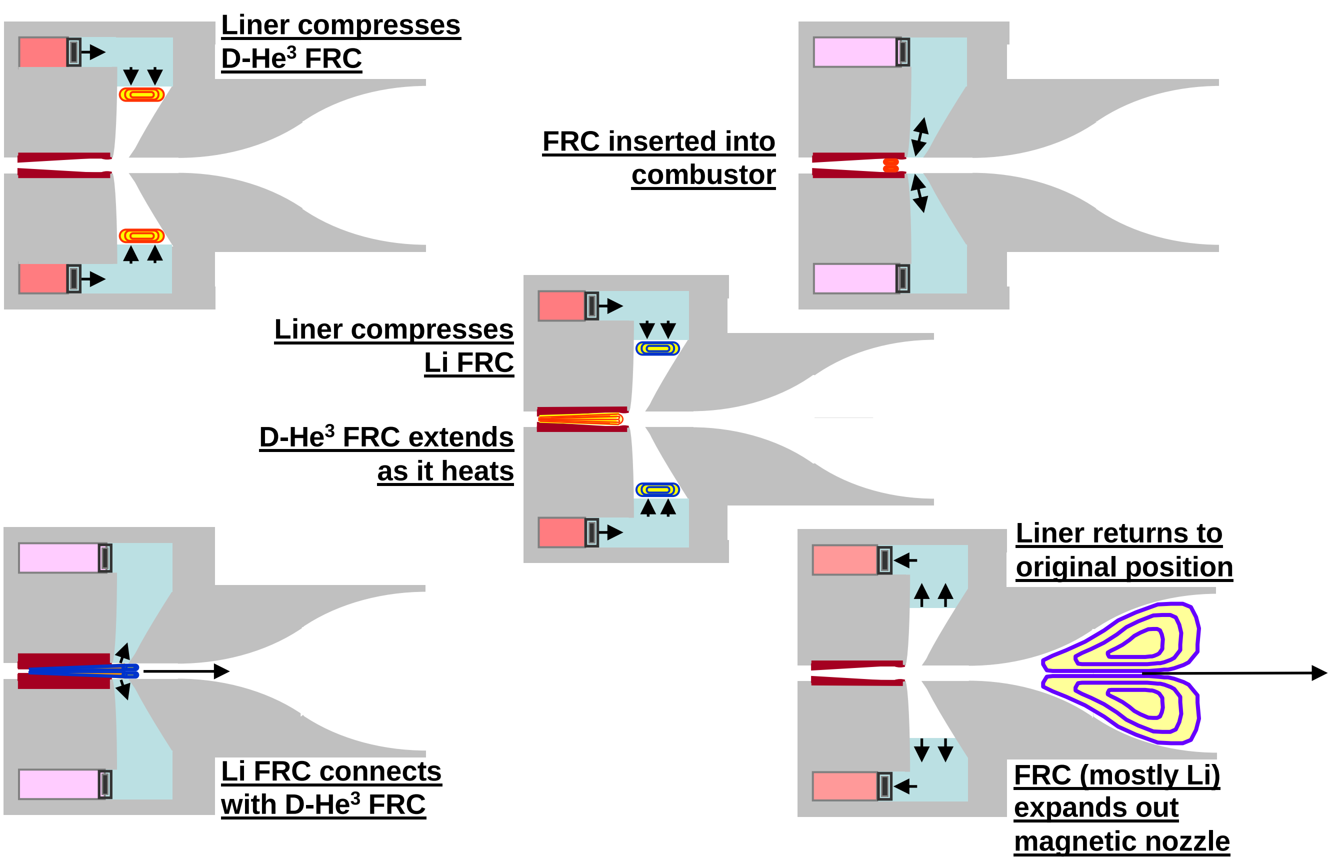

Figure 7. Five stages of liner-driven fusion rocket operation: compression of D-He^3 FRC; insertion into combustor; compression of Li FRC, while burn occurs elongating D-He^3 FRC; merging and extraction of FRCs; FRC exhaust pulse, and re-compression of driver gas.

Figures 6 and 7 display the concept of a liner-driven fusion rocket. As in the liner-driven electric thruster, the stabilized liner compresses the initial FRC to high temperatures adiabatically. The liner then injects the heated FRC into the combustor where deposition of fusion energy causes the FRC to elongate further at constant magnetic pressure. In order to reduce the exhaust speed to values appropriate for a fast trip to the Jovian system, $u_e = \Delta v = 500$ km/s, it is necessary to add more mass to the FRC. This is accomplished conceptually by imploding a second FRC, in this case of lithium, timed to merge with the D-He^3 FRC when its burn is complete. The elongation of the first FRC and its connection with the lithium FRC extracts the merged FRC from the combustor. The pressure of this combined compact toroid provides work to restore the liner to its original energy. As the liner re-compresses the driver gas, the compact toroid expands to high speed borne by the open-field lines of the magnetic nozzle.

V. Sample Design

To provide some quantitative basis for the conceptual design, it is necessary to model the behavior of the FRC, including the associated nuclear energy gain and FRC elongation. The detailed arithmetic of the zero-dimensional model for FRC equilibrium used here is reserved for the Appendix. The basic result is that the proportions of the FRC can determine the relative energies in the plasma and magnetic field. Specification of actual dimensions, initial plasma temperature and the peak operating magnetic field then provides the various system energies and sizes from which a useful set of operating conditions for the fusion rocket are obtained.

In line with the simplification of the formulas by carrying terms only to first-order in the inverse elongation (i.e., plasma diameter small compared to the plasma length), sample values for the several proportions are:

$$r_L/h = 0.1\ ,\quad r_p/l = 0.05\ ,\quad \Delta/r_L = 0.5\ ,\quad \text{and}\quad l/h = 1 - r_L/h$$

where r_L and h are the effective radius and length of the coil surrounding the plasma (as either the liner or the combustor), r_p and l are the (major) radius and length of the plasma, represented as a single-turn coil, and Δ_p is the thickness of the plasma in the radial direction; $r_p + \Delta/2$ is therefore essentially the separatrix radius at mid-length of the plasma. During liner implosion, it is envisioned that the liner length is reduced as the FRC contracts axially. Within the combustor, the effective length of the outer “coil” follows the azimuthal current density induced by displacement of open field-lines onto the conductor surface as the FRC is inserted and elongates. Thus, h/l is taken as constant (along with all radial dimensions) as the FRC heats and extends within the combustor.

As discussed in the Appendix, the internal energy W_p of the plasma is related to the magnetic energy associated with the magnetic flux of the outer “coil” by a dimensionless parameter:

$$\beta_L = (\gamma - 1)W_p/2W_L \tag{1}$$

where, for the D-He^3 plasma

$$W_p = (3/2)\, 5N_DkT \tag{2}$$

The total number of deuterium ions is N_D and the factor of five accounts for an equal quantity of He^3 and the three electrons associated with the ions. The magnetic energy of the outer coil is:

$$W_L = \phi_L^2/2L_L \tag{3}$$

where ϕ_L is the flux associated with the current in the outer coil, the inductance of which is given by:

$$L_L = \mu\pi r_L^2 K_L/h \tag{4}$$

with K_L the correction factor for finite-length coils[11]; in the present case of elongated FRCs, $K_L \approx 0.95$.

The analysis in the Appendix provides that the total magnetic energy W_m (including the self-inductance and mutual inductance energies associated with the inner “coil”) is related to the inductor energy W_L as:

$$W_m = (1 - \beta_L)\, W_L \tag{5}$$

while the plasma energy is:

$$W_p = 3\, \beta_L\, W_L \tag{6}$$

The total energy of the FRC is then:

$$W_T = W_p + W_m$$
$$= (2\beta_L + 1)W_L \qquad (7)$$

The initially specified proportions of the FRC determine the value of β_L, which will be maintained, if poloidal magnetic flux is conserved as a fraction f_p of ϕ_L. The various energies therefore remain in proportion to W_L and increase at fixed radius r_L and ϕ_L as h (and l) increase. Thus, the increase in W_p due to nuclear energy deposition results in increased length of the plasma within the combustor. Note that at constant ϕ_L and r_L, the increase of magnetic energy with length means the magnetic field remains constant. The heat deposition thereby occurs at constant pressure, so the nuclear energy q increases the enthalpy of the FRC plasma as work is done on the confining magnetic fields:

$$q = \Delta H = \gamma \Delta W_p \qquad (8)$$

$$= [2\gamma/(\gamma - 1)]\beta_L \Delta W_L \qquad (9)$$

The change in total energy of the FRC is then:

$$\Delta W_T = (2\beta_L + 1)q/[2\gamma/(\gamma - 1)]\beta_L \qquad (10)$$

The rate of production of nuclear energy is:

$$dq/dt = N_D^2 \langle\sigma v\rangle E_N/V \qquad (11)$$

where V is the effective plasma volume for reactions, $\langle\sigma v\rangle$ is the product of reaction cross-section and relative particle speed integrated over the velocity distribution[12], and E_N is the energy of the charged-particle reaction products (including the side reactions); the total number of helium and deuterium ions are equal. For the FRC elongating from its initial temperature and length to larger values, $V/V_i = T/T_i$, where the subscripts refers to the initial FRC upon injection into the combustor. The nuclear production rate is then:

$$dq/dt = [N_D^2\ kT_i/V_i]\{\langle\sigma v\rangle/kT\}E_N \qquad (12)$$

For kT in the range of 50 – 180 keV, the quantity in braces is constant to about 10% with a value[13] of $9.3\ \text{x}10^{-9}\ m^3/J\text{-}s$. Thus, the fusion burn and FRC elongation occur at a nearly constant rate. The time t_B to achieve a temperature change within the indicated range is based on the enthalpy change of Eq. 8:

$$[N_D^2\ kT_i/V_i]\{\langle\sigma v\rangle/kT\}E_N\ t_B = 5N_Dk(T_f - T_i)\gamma/(\gamma - 1) \qquad (13)$$

so,

$$t_B = 125k(T_f - T_i)/2[5N_D\,kT_i/V_i]\{\langle\sigma v\rangle/kT\}E_N \qquad (14)$$

The quantity in brackets is the initial pressure of the imploded FRC at injection. For E_N = 19.4 MeV (including the charged-particles from the side reactions), a pressure equivalent to a megagauss magnetic field ($4\ \text{x}10^9\ nt/m^2$) and a temperature change of 130 keV, the burn time is 11.3 msec.

The importance of the burn time is that it scales the time for magnetic diffusion into the combustor surface. The associated skin-depth:

$$\delta_L = (\eta_L t_B/\mu)^{1/2} \qquad (15)$$

where η_L is the resistivity of the combustor material, provides the basis for selecting the size of the fusion rocket. Up to this point, only relative dimensions and intensive properties (e.g., temperature, magnetic pressure) have been involved. The loss of electromagnetic energy to resistive heating in the skin-layer must be compensated from the gain of energy due to nuclear reactions. The allowable, fractional loss is proportional to the skin-depth divided by the surface radius, so specification of the acceptable relative loss determines the radius of the combustor from which all other radial dimensions follow. These dimensions, and the choice of FRC elongation, then determine the energy per pulse, and the mass and size of the rocket, including (with specification of the operating frequency) values for the thermal radiators.

The FRC length in the combustor increases linearly with time from its initial length l_i, so the skin-depth varies parabolically from zero at the most upstream end to the value provided by Eq. 14 (which also applies along the initial length at insertion). The energy lost to resistive heating during the burn-time, and the subsequent, very rapid extraction of the D-He^3 FRC from the combustor, is approximately the magnetic energy in the skin-layer volume; (during the field rise, half this energy is lost to heat, but an equal amount is lost when the surface field returns to low values.) In terms of the inductor energy W_L, the relative energy consumed by resistive heating is:

$$w_R = (B^2/2\mu)\, V_{skin}/W_L \tag{16}$$

with, to first-order, $W_L = (B^2/2\mu)\pi r_L^2 l_f$ and

$$V_{skin} = \pi\{[2r_L\delta_L + \delta_L^2]l_i + [(4/3)r_L\delta_L + \delta_L^2/2](l_f - l_i)\}$$

so,

$$w_R = \{[2\delta_L/r_L + (\delta_L/r_L)^2](l_i/l_f) + [(4/3)\delta_L/r_L+(\delta_L/r_L)^2/2](1 - l_i/l_f)\} \tag{17}$$

With the ratio of initial to final length (l_i/l_f) set to the ratio of the previous range of temperature values, T_i/T_f = 50/180 = 0.28, a skin-depth that is 10% of the combustor radius would correspond to an energy loss w_R = 15.8% of the inductor energy W_L. From Eq. 7, with $\beta_L = 0.25$, this represents 10.5% of the total energy of the D-He^3 FRC. For the same temperature ratio (which corresponds to the burn-time), the gain in total energy:

$$\Delta W_T = (2\beta_L + 1)W_{Lf}(1 - T_i/T_f) \tag{18}$$

is reduced by 14.6%. This net energy is delivered to the pulsed rocket exhaust. The initial energy W_{Ti} of the D-He^3 FRC and that of the compressed Li FRC are available to return work to the liner implosion system.

The heat generated in the combustor skin-layer and that due to neutrons from the plasma must be processed by the thermal radiators. Both these energies are proportional to the change in energy ΔW_T, with the neutron energy equal to a fraction f_n of the heat q:

$$\begin{aligned} q_n &= f_n q \\ &= f_n[2\gamma/(\gamma - 1)]\beta_L\Delta W_T/(2\beta_L + 1) \end{aligned} \tag{19}$$

In the vicinity of T = 100 keV, for a pressure corresponding to a megagauss field, the characteristic reaction time for a tritium nucleus in the deuterium plasma is about 25 msec. Only a fraction (~1%) of the neutron energy (14.1 MeV) from the D-T reaction is available during the burn-time of 11.3 msec. If all of the 2.45 MeV neutrons from the D-D reactions are included, an estimate for f_n is 1.2%. Thus, with $\beta_L = 0.25$, the heat that must be rejected is H_R = 15.6% ΔW_T, (neglecting for now the much lower fractional losses associated with the liner implosion process).

To proceed, it is necessary to make some estimate of the size and mass of the system. With a burn-time of 11.3 msec at a peak magnetic field of a megagauss, the skin-depth in a combustor of copper is about 2.7 cm. (At the peak surface field of about a megagauss, the diffusion process is nonlinear[14] and the heated copper has a resistivity of about 8 $\times 10^{-8}\Omega$-m.) From the earlier specification, the radius is then r_L= 27 cm. If the elongation at the end of the burn is $l_f/2(r_p + \Delta/2) = 12$, then the l_f = 4.9 m. The initial length of the FRC is $l_i = l_f(T_i/T_f)$ = 1.4 m, so the FRC, before a radial compression of $\alpha = 10$, would have a length of $l_i\alpha^{0.4}$ = 3.4 m; the initial radius r_o of the inner surface of the liner would be 2.7 m.

From the conceptual view of Fig. 6, a rough estimate of the mass of the rocket engine may be attempted. For example, a total length l_T is at least the sum $l_f + l_i + l_N$ = 12.6 m, if the length of the nozzle l_N is taken as equal to that of the combined FRCs. With an additional 20%, the rocket system length is then 15.1 m. The volume of the lithium liner that must be displaced for the assumed compression, following the contraction of the FRC is ΔV_L = 60.6 m^3. If the difference in radius across the piston face is half the original liner length, Δr_D = 1.7 m, then the stroke of the piston to provide ΔV_L is Δz_D = 1.6 m.

To minimize the drive pressure for a given volume change, it is useful to have the largest initial volume. This may be accomplished by reducing the radius of the driver-gas volume to the vicinity of the combustor (i.e., taking up more of the structural volume shown in Fig. 6 with the open space for the driver-gas). The actual dimensions will depend on the allowable mechanical stress, which requires some estimate of the energies required. The initial energy of the FRC is known from the specified operating magnetic field and the dimensions previously stated:

$$W_{Ti} = (2\beta_L + 1)\,(B^2/2\mu)\,\pi r_L^2 l_i = 1.92\ \text{GJ} \tag{20}$$

Compressibility effects and the need for rotational stabilization increase the amount of work that must be supplied above this energy by a factor of about 5 for a radial compression ratio of 10:1, allowing for axial converging effects not included in the earlier cylindrical analysis[9]. Some additional energy is also required for the outer amount of lithium at the radius of the piston-face, which must make the turn for the radial implosion. This increases the necessary energy by about 43%. The total work that must be supplied by the driver gas is then $W_D = 13.8$ GJ. If the length of the initial gas volume is taken as $l_f - \Delta z_D$, (arbitrarily in order to fit within the length indicated in Fig. 6), and the inner radius of this volume is twice the outer radius of the combustor, $r_I = 2 \times 29.7$ cm = 60 cm, then the gas volume is $V_i = 197\ m^3$. The initial gas pressure to supply the required work is then 2895 atm, which is equivalent to the pressure of a magnetic field of 26.9 T (similar to the field on the high-strength aluminum driver coil for the imploding liner experiments[2] on the Suzy II capacitor bank at NRL). For the same material (with yield strength of 706 kpsi) invoked to enable the combustor, a total outer radius $r_T = 5$ m for the implosion system would allow operation at less than 50% of yield strength. The basic size of the rocket engine is thus about 10m in diameter and 15 m long. Simple geometric estimates of the volumes occupied by structural material (at an average density of 2.2 g/cm^3 vs the actual density of 1.56 g/cm^3 for the high-strength dielectric), the lithium liner material (0.47 g/cm^3) and the gas or vacuum (~0.0 g/cm^3) provide a total mass of 950 t, which may be readily rounded up to 1000 t.

For the previously estimated initial energy for the FRC, the energy available for the pulsed rocket exhaust is:

$$\begin{aligned} W_{ex} &= \{1 - 14.6\%\}\Delta W_T \\ &= 0.854\,[T_f/T_i - 1]W_{Ti} \\ &= 4.2\ \text{GJ} \end{aligned} \tag{21}$$

The associated heat for rejection is $H_R = 0.78$ GJ. The mass of the thermal radiators will depend on the frequency of pulse repetition ν, which depends on satisfying the desire for a minimum acceleration of $0.01g_o$.

The earlier paper[1] used a simple kinematic analysis of the transit from the Moon to Jupiter at constant acceleration to cover the separation distance in two phases: acceleration a to $\Delta v/2$, followed by deceleration, also at a, to Jupiter's orbital speed. For the present design concept, the jet power is fixed at $P_J = W_E\nu$, so constant acceleration as the total mass M_T decreases means that the mass flow rate, $w = -\,dM_T/dt$, must be reduced during the trip, where:

$$P_J = wu_{ex}^2/2 = M_T^2a^2/2w \tag{22}$$

so,

$$-dM_T/dt = M_T^2(a/u_o)/M_o \tag{23}$$

where M_o is the total mass at the start of the trip when $u_{ex} = u_o$.

The reduction in w may be accomplished by providing Li FRCs of decreasing mass, which results in higher value of exhaust speed. Solution of Eq. 23 provides the masses of the payload M_{pay}, the engine M_E, radiators M_R and tankage M_t delivered to the desired Δv, starting with an initial exhaust speed u_o and initial mass $M_o = M_{pay} + M_E + M_R + M_t + M_{prop}$:

$$(M_{pay} + M_E + M_R + M_t)/(M_{pay} + M_E + M_R + M_t + M_{prop}) = 1/(1 + \Delta v/u_o) \tag{24}$$

where M_{prop} is the initial mass of the propellant, and the mass of the tankage can be taken as a fraction of the propellant mass $f_t M_{prop}$. With $u_o = \Delta v$, and $f_t = 0.1$, the propellant mass is:

$$\begin{aligned} M_{prop} &= (M_{pay} + M_E + M_R)(\Delta v/u_o) /(1 - f_t \Delta v/u_o) \\ &= 1.11\ (M_{pay} + M_E + M_R) \end{aligned} \tag{25}$$

So the total initial mass is:

$$M_o = 2.22\ (M_{pay} + M_E + M_R) \tag{26}$$

Let the payload mass equal a fraction f_V of the engine plus radiator mass. Then

$$M_o = 2.22\ (1 + f_V)(M_E + M_R) \tag{27}$$

The jet power is:

$$P_J = M_o a u_o/2 = 2.22\ (1 + f_V)(M_E + M_R) a u_o/2 \tag{28}$$

But, $P_J = W_{ex}\nu$, and $M_R = H_R\nu/\alpha_R$, where α_R is the specific power of the thermal radiator system. Therefore, the payload fraction is:

$$f_V = W_{ex}\nu/[1.11\ a u_o(M_E + H_R\nu/\alpha_R)] – 1 \tag{29}$$

If $\alpha_J = 50$ kw/kg and $\nu = 20$ Hz, the mass of the radiators is $M_R = 312$ t, and the payload mass fraction would then be $f_V = 15.4\%$. This corresponds to a payload delivered to the Jovian system of 202 t at an initial thrust level of 336,000 nt (76,400 lbf). The total radiator area would be 312,000 m^2, which could consist of six panels (two-sides each of a three-finned structure) 177 m x 294 m. The payload mass is about half that of the International Space Station (450 t), which is distributed over 72.8 m x 108.5m.

VI. Interstellar

The preceding calculations have been done in the context of a fast voyage to the Jovian system with continual acceleration of 1%g_o. The exhaust speed for this mission is considerably less than the particle speeds available in the D-He^3 FRC, so a second FRC of lithium is added with (initially) about 250 times more mass; M_T decreases by a factor of two by the end of the transit, so the lithium factor is reduced to 125. Suppose instead, in order to achieve the maximum exhaust speed, no lithium is added. The mass of the D-He^3 FRC is $5m_pN_D$, where m_p is the proton mass and the total number of deuterium ions is:

$$N_D = (2/5)\beta_L W_{Ti}/(2\beta_L + 1)kT_i \tag{30}$$

With the exhaust energy from Eq. 20, the exhaust speed of the D-He^3 plasma is:

$$\begin{aligned} u_{ex} &=\{(0.856)(2\beta_L + 1)[T_f/T_i – 1](kT_i/m_p)/\beta_L\}^{1/2} \\ &= 8 \times 10^6 \text{ m/s} \end{aligned} \tag{31}$$

The mass flow rate is now constant with D-He^3 as the propellant, and the mass fraction delivered to the desired Δv results from the usual rocket equation:

$$(M_{pay} + M_E + M_R + M_t)/M_o = \exp -\Delta v/u_{ex} \tag{32}$$

The propellant mass is then initially:

$$M_{prop} = (1 + f_V)(M_E + M_R)[\exp(\Delta v/u_{ex}) – 1]/(1 - f_t [\exp(\Delta v/u_{ex}) – 1]) \tag{33}$$

where the importance of very low tankage mass fraction f_t is evident as limiting the possible Δv. The time for a mission to a nearby star comprises the time to attain Δv and the time then to coast at this speed for most of the journey. The former time equals the propellant mass divided by the mass flow rate:

$$- dM_{prop}/dt = 2W_E\nu/u_{ex}^2 \quad (34)$$

At a repetition rate of 40 Hz, appropriate to compression of a single FRC, a tankage mass fraction f_t = 5%, and reducing the mission to a “fly-by” with $f_V \approx 0$, the acceleration time is 41.3 years for $\Delta v/u_{ex}$=1.5, and the total trip time to Alpha Centauri (4.3 light-yrs) is 149 years. Thus, even with an advanced fusion fuel, it is unlikely to reach the nearest star within a normal NASA career. A more adventurous extrapolation could examine the possibility of burning the D-He^3 fuel more toward completion, resulting in an FRC with a much higher population of particles at a few MeV per nucleon. An improvement in exhaust speed by a factor of a few ($u_{ex} \approx 0.1c$) could reduce the trip time to under a century. In any event, it may be necessary to accept instead the so-called “Long Trip”, involving multiple generations of crew in space over the course of the voyage. In this case, the spaceship must be very large compared to even the previously calculated mission to Europa. Within present understanding, the very large sizes and powers associated with fusion could be quite appropriate technology.

VII. Power Production

It is useful to note that the same pulsed, high speed plasma developed for space propulsion could be used to provide electrical power. This is indicated conceptual in Fig. 8, based on a notion called Pulsar explored in the late seventies[15].

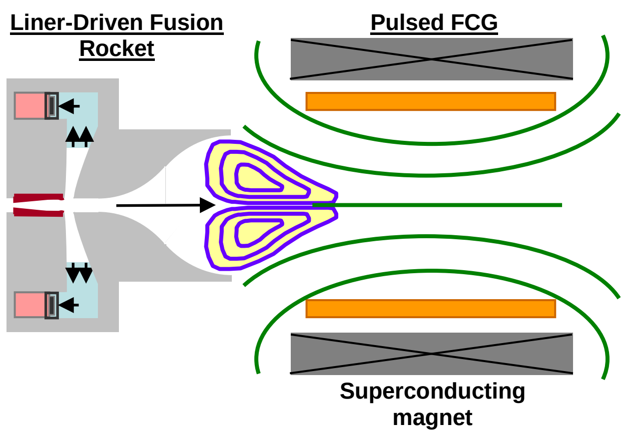

Figure 8. Concept of electrical power production using the pulsed exhaust from the liner-drive fusion rocket to displace/compress magnetic flux from a superconducting magnetic relative to a normal coil.

When the payload and propulsion system reach their destination, electrical power may be needed at levels much larger than employed during the transit. Such power could, of course, be available from pre-positioned resources (e.g., fission or RTG power supplies, communication equipment, propellant) sent earlier by low-thrust electric propulsion. The present rocket, however, offers a gigawatt-level electrical source that might enable some explorations.

In principle, the same arrangement depicted in Fig. 8 has terrestrial applications, if competitive with more mundane sources of electrical power. Without the constraints posed by space propulsion requirements, various factors, such as the repetition rate could be relaxed, in order to satisfy ground-based needs (e.g., lower output power). The purported availability of He^3 on the Moon suggests that such a system could best find application above the gravity-well of the earth. Without the thermal rejection burdens in space, technology discussed here (and in previous Linus reactor schemes[8]) could return to the more accessible D-T reactions.

VIII. Concluding Remarks

This paper has examined several aspects of a fusion propulsion system aimed at very high-energy missions to the outer planets and beyond. Much work would need to be done before a detailed design could be attempted. In particular, the use of stabilized imploding liners to compress plasma was encouraging, but embryonic at the time the Linus program closed at NRL. That program ended, in large part, because of the lack of a satisfactory plasma target. While many studies have been done in the intervening years on FRCs, these have been insufficiently funded and sustained to resolve many questions about stability and transport, especially at the very high temperatures needed for advanced-fuel reactions. Indeed, it appears, given the complexities of FRC formation and operation, that progress will require an empirical approach, (i.e., experimental exploration to find useful conditions, followed by theoretical modeling to explain success). Private companies are pursuing liquid metal implosions and plasma targets[16] with venture capital. It is unlikely, however, that substantial government funding will occur in the foreseeable future, given the present commitments to ITER (formerly aka, the International Thermonuclear Experimental Reactor). Fortunately, in some sense, the timescale for this commitment matches that for generating the will and resources for long-range, high-energy, crewed space missions. Thus, perhaps by the middle of the 21st century, sufficient understanding, resources and interest will have accumulated to enable human exploration of the Solar System in a style heretofore only seen in science fiction.

References

[1] Turchi, P.J., “Advanced Electric Propulsion Concepts Based on Magnetic Flux Compression and Expansion,” in Proc. of 29th Int. Electric Propulsion Conf., Princeton (2005). Paper IEPC-2005-049.

[2] Turchi, P.J. et al, “Review of the NRL Liner Implosion Program,” in Megagauss Physics and Technology, P.J. Turchi, ed., Plenum, NY (1980). P. 375.

[3] Es’kov, A.G. , Kurtmullaev, R.Kh., et al, “Liner Compression of a Toroidal High-β Plasma,” in Pulsed High Beta Plasmas, D.E. Evans, ed., Pergamon, NY (1976). P.489.

[4] Tuszewski, M., “Field Reversed Configurations,” Nucl. Fusion, 28, 11, 2033 (1988).

[5] Steinhauer, L.C., “Review of Field-Reversed Configurations”, Phys. of Plasmas, 18, 070501 (2011).

[6] Turchi, P.J., “Imploding Liner Compression of Plasma: Concepts and Issues,” IEEE Trans. on Plasma Science, 36, 1, 52 (2008).

[7] Stuhlinger, E., Ion Propulsion for Spacefight, McGraw-Hill, NY (1964). P. 75.

[8] Turchi, P.J.,"A Compact-Toroid Fusion Reactor Design at 0.5 Megagauss, Based on Stabilized Liner Implosion Techniques", in Ultrahigh Magnetic Fields, V.M. Titov and G.A. Shvetsov, eds., Nauka, Moscow (1984). P. 184.

[9] Book, D.L. and Turchi, P.J., "Dynamics of Rotationally Stabilized Implosions of Compressible Liquid Shells,” Phys. Fluids 22, 68 (1979).

[10] Sims, J.R., et al, “Assembly, Commissioning and Operation of the NHMFL 100 Tesla Multi-Pulse Magnet System,” IEEE Trans. on Applied Superconductivity, 18, 2, 587 (June 2008).

[11] Knoepfel, H., Pulsed High Magnetic Fields, North-Holland, NY (1970). P. 317.

[12] Glasstone, S. and Lovberg, R.H., Controlled Thermonuclear Reactions, Van Nostrand, NY (1960). P. 19.

[13] Cox,L.T., Jr., “Thermonuclear Cross Section and Reaction Rate Parameter Data Compilation,” Air Force Phillips Laboratory Report AL-TR-90-053, July 1991. P.53-56.

[14] Turchi, P.J., “Physical Evolution of Metal Surface Layers Exposed to Pulsed Megagauss Magnetic Fields,“ in *2012 14th International Conference on Megagauss Magnetic Field Generation and Related Topics (MEGAGAUSS)*, Kiuttu, G.F., Struve, K.W., Degnan, J.H., eds. (2014). IEEE Xplore CD, www.ieee.org/web/publications/xplore/.

[15] Cowan, M.L., et al, “Pulsar – A Field Compression Generator for Pulsed Power,” Proc. of 6th Symp. on Engineering Problems of Fusion Research, CH1 097-5NPS. P. 308 (1976).

[16] Richardson, D., et al, “Status of Progress Towards Acoustic Magnetized Target Fusion at General Fusion,” 34th Annual Conf. of Canadian Nuclear Society, 9-12 June 2013. Toronto, Canada.

[17] Turchi, P.J., “Electric Rocket Propulsion Systems,” in Space Propulsion Analysis and Design, Humble, R.W., Henry, G.N., Larson, W.J., eds., McGraw-Hill, NY (1995). P. 544.

[18] Milroy, R.D. and Steinhauer, L.C., “Toroidal field stabilization of the rotational instability in field-reversed configurations,” Phys. of Plasmas, 15, 022508 (2008).

[19] Handbook of Chemistry and Physics, The Chemical Rubber Publishing Co., Cleveland, 44th Ed. (1963). P. 3399.

Appendix

Zero-Dimensional Model for FRC

To provide some basis for the relationship of FRC dimensions to plasma and magnetic energy densities, it is useful to employ at least a zero-dimensional model. Such a model may be constructed from the forces acting on the FRC, if treated in the manner of coils carrying current. The electromagnetic forces between two circuits '1' and '2' with currents J_1 and J_2 may be written as[17]:

$$\mathbf{F}_1 = (J_1^2/2)\,\nabla L_1 + J_1 J_2 \nabla M_{12} \quad \text{and} \quad \mathbf{F}_2 = (J_2^2/2)\,\nabla L_2 + J_1 J_2 \nabla M_{12} \qquad \text{(A1)}$$

where the gradient operations are performed in the direction of variation of the geometric dimensions of the self-inductances $L_1 = L_1(r_1, h_1)$ and $L_2 = L_2(r_2, h_2)$, and the mutual inductance $M_{12} = M_{12}(r_1, h_1, r_2, h_2)$. The currents may be expressed in terms of the magnetic flux values ϕ_1 and ϕ_2, so:

$$\mathbf{F}_1 = (\phi_1^2/2L_1^2)\,\nabla L_1 + (\phi_1\phi_2/L_1L_2)\nabla M_{12} \quad \text{and} \quad \mathbf{F}_2 = (\phi_2^2/2L_2^2)\,\nabla L_2 + (\phi_1\phi_2/L_1L_2)\nabla M_{12} \qquad \text{(A2)}$$

For example, the force to change the radius r_1 of circuit '1' is:

$$F_{r1} = (\phi_1^2/2L_1^2)\,\partial L_1/\partial r_1 + (\phi_1\phi_2/L_1L_2)\,\partial M_{12}/\partial r_1 \qquad \text{(A3)}$$

and is exerted in the direction to increase r_1. For the basic FRC, the currents are in the azimuthal (aka, toroidal) direction and produce forces in the rz-plane. These currents travel in the surface of the metal surrounding the FRC, e.g., the imploding liner or the combustor, and in the plasma. The plasma may also carry poloidal current in the rz-plane, associated with toroidal magnetic field. Such toroidal fields may arise naturally in the FRC or by design, in order to enhance stability[18]. The total electromagnetic force on the circuit elements can therefore have an additional component:

$$\mathbf{F}_T = (\phi_T^2/2L_T^2)\,\nabla L_T \qquad \text{(A4)}$$

where the subscript 'T' refers to the presence of toroidal field. For example, with $L_T = (\mu/2\pi) l \ln(r_{p2}/r_{p1})$, the axial electromagnetic force on plasma with poloidal current J_p traveling in a loop of radii r_{p1} and r_{p2} is $F_h = (\mu/4\pi)J_p^2 \ln(r_{p2}/r_{p1})$, which is added to the axial electromagnetic forces in the direction of the plasma length obtained from Eq. A3. Note that both self- and mutual forces can be involved in attempting to change l.

For the FRC within the liner/combustor (Fig. A-1), self-inductances are given by:

For the liner/combustor

$$L_L = \mu\pi r_L^2 K_L/h \qquad \text{(A5)}$$

For the plasma

$$L_p = \mu\pi r_p^2 K_p/l \qquad \text{(A6)}$$

where the subscripts 'L' and 'p' refer to liner/combustor and plasma, respectively, and K_L and K_p are correction factors for finite-length coils. With the length h of the current distribution in the liner/combustor following the length l of the plasma, and both lengths large compared to the liner and plasma (major) radii, r_L and r_p, respectively, both correction factors are slightly less than unity[11]. The inductance associated with the toroidal field is:

$$L_T = (\mu/2\pi) l \ln[(r_p + \Delta/2)/(r_p - \Delta/2)] \qquad \text{(A7)}$$

where the radial thickness of the plasma is Δ.

The mutual inductance between two coils (the outer, the longer) is given by[19]:

$$M_{Lp} = \{\mu\pi r_p^2/[h^2 + 4r_L^2]^{1/2}\}\,\{1 - 2r_p^2 r_L^2 (l^2/r_p^2 - 3)/[h^2 + 4r_L^2]^2\} \qquad \text{(A8)}$$

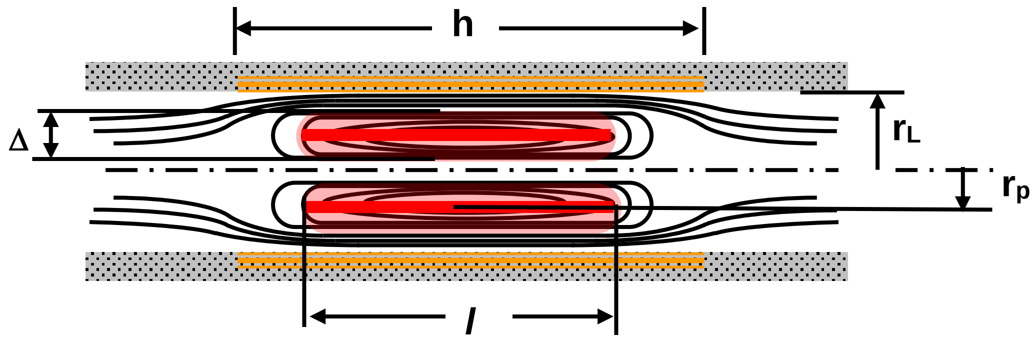

Figure A-1. Dimensions of equivalent inner and outer coils to model FRC in liner/combustor of Fig. 3.

In addition to the electromagnetic forces, there are components of the force due to the plasma pressure, p, which may be written in terms of the total plasma energy W_p and the plasma volume V:

$$p = (\gamma - 1)\, W_p/V \tag{A9}$$

where $\gamma = 5/3$ is the ratio of specific heats and

$$W_p = 5N_D kT/(\gamma - 1) \tag{A10}$$

The factor of 5 is for the two ions (D and He) and their three electrons. In line with the electromagnetic force, the force due to plasma pressure may be written as:

$$\begin{aligned} \mathbf{F}_p &= -\nabla W_p \\ &= (\gamma - 1)W_p[(1/V)\nabla V] \end{aligned} \tag{A11}$$

The three equations for the FRC equilibrium within the liner/combustor are:

Axial (l-direction)

$$(\gamma - 1)W_p/l + (\phi_p^2/2L_p^2)\, \partial L_p/\partial l + (\phi_T^2/2L_T^2)\, \partial L_T/\partial l + (\phi_L\phi_p/L_L L_p)\, \partial M_{Lp}/\partial l = (B_E^2/2\mu)2\pi r_p\Delta \tag{A12}$$

where β is a profile parameter and B_E is the magnetic field between the ends of the FRC and the curved wall of the combustor used to retain the FRC axially.

Major radial (r_p-direction)

$$(\gamma - 1)W_p/r_p + (\phi_p^2/2L_p^2)\, \partial L_p/\partial r_p + (\phi_T^2/2L_T^2)\, \partial L_T/\partial r_p + (\phi_L\phi_p/L_L L_p)\, \partial M_{Lp}/\partial r_p = 0 \tag{A13}$$

Minor radial (Δ-direction)

$$(B^2/2\mu)2\pi r_p l = (\phi_T^2/2L_T^2)\, \partial L_T/\partial\Delta + (\gamma - 1)W_p/\Delta \tag{A14}$$

where $$(B^2/2\mu)2\pi r_p l = (\phi_L^2/2L_L^2)\, \partial L_L/\partial r_L + (\phi_L\phi_p/L_L L_p)\, \partial M_{Lp}/\partial r_L \tag{A15}$$

It is useful to normalize these equations by dividing all magnetic fluxes by ϕ_L and the plasma energy by that of the inductor L_L, so:

$$\beta_L = [(\gamma - 1)W_p/2]/[\phi_L^2/2L_L] \quad , \quad f_p = |\phi_p/\phi_L| > 0 \quad , \quad f_T = \phi_T/\phi_L \tag{A16}$$

The force equations then become:

Axial

$$\beta_L/l + f_p^2(L_L/L_p)[(1/2L_p)\,\partial L_p/\partial l] + f_T^2(L_L/L_T)[(1/2L_T)\partial L_T/\partial l] - f_p[(1/L_p)\partial M_{Lp}/\partial l] = f_E L_L/l \qquad \text{(A17)}$$

Note that there is a minus sign at the mutual inductance force term because $f_p > 0$ and $\phi_p\phi_L < 0$; the azimuthal currents in the liner/combustor and the plasma are oppositely directed. The term on the right-hand side expresses the small end forces that help stabilize the FRC position, (and will be neglected here).

Major radial

$$\beta_L/r_p + f_p^2(L_L/L_p)[(1/2L_p)\,\partial L_p/\partial r_p] + f_T^2(L_L/L_T)[(1/2L_T)\partial L_T/\partial r_p] - f_p[(1\,/L_p)\partial M_{Lp}/\partial r_p] = 0 \qquad \text{(A18)}$$

Minor radial

$$\beta_L/\Delta + f_T^2(L_L/L_T)[(1/2L_T)\partial L_T/\partial \Delta] = \{(1/2L_L)\,\partial L_L/\partial r_L - f_p[(1\,/L_p)\partial M_{Lp}/\partial r_L]\} \qquad \text{(A19)}$$

With the inductance formulas A5-8, the preceding equations may be expanded in order to solve for β_L, f_p and Δ, or some combination with the several ratios of radii and lengths. Unfortunately, the general problem involves algebraic equations at the level of cubic and higher, so some sort of iterative procedure would be required. Instead, if solution is restricted to the situation of high elongation e = plasma length divided by separatrix diameter, terms higher order than e^{-1} can be dropped and the resulting equations solved directly. (In the same spirit, if the toroidal magnetic field is provided for stability purposes, with a value 30% or less than the peak poloidal field[18], the associated force term may also be ignored.)

The two equations for β_L and f_p become:

$$\beta_L = f_p^2(r_L/r_p)^2(K_L/K_p)(l/h)/2 \qquad \text{(A20)}$$

and

$$\beta_L + f_p^2(r_L/r_p)^2(K_L/K_p)(l/h) = 2f_p(l/h)/K_p \qquad \text{(A21)}$$

from which

$$f_p = (4/3)\,(r_p/r_L)^2/K_L \qquad \text{(A22)}$$

and

$$\beta_L = (8/9)\,(r_p/r_L)^2/K_L K_p \qquad \text{(A23)}$$

With $r_p/r_L = 0.5$, $l \approx h$, and $K_L = K_p = 0.95$, $f_p = 0.351$ and $\beta_L = 0.246$. In terms of the inductor energy $W_L = \phi_L{}^2/2L_L$, the plasma energy is $3\beta_L W_L$ or about 75% of the inductor energy. The total magnetic energy is:

$$\begin{aligned} W_m &= \phi_L{}^2/2L_L + \phi_p{}^2/2L_p + (\phi_L\phi_p/L_L L_p)M_{Lp} \\ &= W_L[\,1 + f_p^2(r_L/r_p)^2(K_L/K_p) - 2f_p M_{Lp}/L_p\,] \\ &= 75.4\%\ W_L \end{aligned} \qquad \text{(A24)}$$

Thus, the zero-dimensional model offers $W_p \approx W_m$, which represents an efficiency of utilization of the total FRC energy of $W_p/(W_p + W_m) \approx 50\%$. Substitution of Eq. A21 in Eq. A24 provides the magnetic energy:

$$W_m = (1 - \beta_L)W_L \qquad \text{(A25)}$$

so the total energy of the FRC in the liner/combustor is: $W_T = (2\beta_L + 1)W_L$.

The minor radial equation, to the same order of approximation, provides $\Delta/r_L = 2\beta_L \approx 0.5$. Note that at this order, Δ does not affect the other results.