The role of the electron energy balance in Hall thruster plasma instabilities

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Abstract: Using the fluid equations of Hall thruster plasma we analyze the influence of the electron energy balance on the stability of ion sound modes. For sufficiently low frequencies the gains and losses in the source term are approximately equal, thus the temperature can be in principle determined in terms of other dependent variables. This permits to reduce the umber of equations. It appears however, that the new system can have in some regions complex characteristics. This in turn implies instability of certain modes with frequencies lower than some critical frequency.

Nomenclature

β = ionization energy for Ex.
e = electron charge
E_k = electron energy
eI = total current density
m_i = ion mass
m_e = electron mass
N_a = density of neutral atoms
n_i = ion density
V_a = neutral atoms (axial) velocity
V_i = ion velocity

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T_e = electron temperature
ω_B = electron cyclotron frequency
ν_{eff} = effective electron collision frequency for the momentum transfer
ν_e = electron collision frequency for the energy transfer
V_e = axial electron fluid velocity
$V_{e\theta}$ = azimuthal electron fluid velocity

I.Introduction

One of the most challenging problems in the theory of plasma is to determine effectively the conditions for the appearance of turbulence and numerous instabilities observed in experiments. Different methods are known and used. One of the less known was sketched in the book of Witham [1] in his discussion of wave hierarchies. We slightly develop here his ideas. To explain our approach let us start with a simple example of a system of two linear equations with constant coefficients

$$\begin{aligned} u_t + u_x + v_x &= 0 \\ v_t + av_x &= \mu\,(ru - v) \end{aligned} \qquad (6)$$

whose matrix $\begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix}$ has two distinct eigen-values if $a \neq 0$. Consequently the system is strictly hyperbolic. It has two characteristics i.e. the short waves can propagate either with speed $c_1 = 1$ or $c_2 = a$. Let us notice also that $1/\mu$ defines here the characteristic time scale in which the influence of RHS is manifested. If μ is large and positive one expects that for perturbations of large enough length, or better: of low enough frequency, $\omega < \mu$, the right hand side of (6) is approximately equal to zero,

$$v = ru + O(\omega/\mu) \qquad (7)$$

Using this relation in the first equation, one obtains in the limit $1/\mu \rightarrow 0$, the reduced system

$$u_t + (1 + r)u_x = 0 \qquad (8)$$

which is a long wave approximation of system (6). Eq. (8) suggests that one may expect existence of a "lower order waves" propagating with the speed $c_0 = 1 + r$. The approximation (8) makes sense however, only if the new characteristic speed $c_0 = 1 + r$ lies between the characteristic speeds of the original system (6): $c_1 < c_0 < c_2$ (if $c_1 < c_2$). Indeed, perturbations satisfying Eqs (6) cannot move slower than c_1, and faster than c_2. Therefore if $c_0 \notin [c_1, c_2]$, then at least one of the waves of Eqs (6) must be unstable in the range of high frequencies and moreover, the growth of instability should have the scale comparable with $1/\mu$. If it would be longer, then Eq. (8) could still be a good approximation on some time scale. This however would contradict causality. Let us note that this reasoning is not limited to linear equations.

In case of a system consisting of three or more equations the analysis becomes more complex and more possibilities can be encountered. Let us take for example the following hyperbolic system

$$\begin{aligned} u_t^1 + u_x^2 + v_x &= 0 \\ u_t^2 + u_x^1 &= 0 \\ v_t + av_x &= \mu\,(ru^2 - v) \end{aligned} \qquad (9)$$

The long wave approximation (i.e. low frequency, $\omega < \mu$) leads in this case to

$$\begin{aligned} u_t^1 + (1 + a)u_x^2 &= 0 \\ u_t^2 + u_x^1 &= 0 \end{aligned} \qquad (10)$$

whose characteristic speeds are $C_1 = -\sqrt{1+r}$, $C_2 = \sqrt{1+r}$, whereas characteristic speeds of system (9) are $c_1 = -1$, $c_2 = 1$, $c_3 = a$. Now we have two possibilities:

1. If $1+r \geq 0$ - then system (10) is hyperbolic
2. If $1+r < 0$ - then the system is elliptic.

In the first case we can speak of new, lower order, families of waves, provided however, that the following stability condition

$$c_1 \leq C_1 \leq c_2 \leq C_2 \leq c_3$$

is fulfilled. If this condition is not satisfied then some of short wave modes, of original system (9) are unstable and because of that the "lower order waves" (with velocities C_1, C_2) cannot be formed. Obviously, in such a case Eqs (10) is not a good approximation of (9).

In the second case, $1+r<0$, we have complex characteristics, hence the Cauchy problem for Eqs (10) is ill posed: one encounters the Hadamard instability – increment of growth is increasing when frequency increases -shorter the wave more unstable it is! Having in mind however that (10) was derived from (9) under the assumption that solution consists mostly of low frequency waves, we can not expect that the conclusion concerning the instability increment is true for waves of frequencies higher than μ ore close to μ. Thus μ is a critical frequency and it defines three scales of frequencies $\omega \gg \mu$, $\omega = O(\mu)$, $\omega \ll \mu$. Let us note also that the derivative of the RHS of the third equation of Esq. (9) is equal to $-\mu$, hence it is negative, which is also a sort of stability condition, allowing one to derive Esq. (7) at least in the range of low frequencies.

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II. The flow equations

Under the plasma quasi-neutrality condition, $n_i = n_e$, the simple fluid description of the Hall thruster plasma uses the following hyperbolic system of four equations [1,2,3]

$$\frac{\partial N_a}{\partial t} + V_a \frac{\partial N_a}{\partial x} = -\beta N_a n_i \tag{1}$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial (n_i V_i)}{\partial x} = \beta N_a n_i \tag{2}$$

$$\frac{\partial V_i}{\partial t} + V_i \frac{\partial V_i}{\partial x} + \frac{1}{n_e}\frac{\partial}{\partial x}\left(\frac{T_e}{m_i} n_e\right) = -\nu_{eff}\left(\frac{I(t)}{n_i} - V_i\right) + \beta N_a (V_a - V_i) \tag{3}$$

$$\frac{2}{3\sqrt{T_e}}\left\{\frac{\partial}{\partial t}(T_e^{3/2}) + \frac{\partial}{\partial x}(V_e T_e^{3/2})\right\} = \tfrac{2}{3} Q, \tag{4}$$

for N_a- the density of neutrals, n_i- ion density (=electron density), V_i – axial ion velocity and T_e – electron temperature in energy units. $\beta = \beta(E_k + \frac{3}{2}T_e)$ is the ionization rate of Xenon. Here E_k is the energy of the "ordered motion" of electrons

$$E_k = \frac{m_e}{2}(V_e^2 + V_{e\theta}^2) = \frac{m_e}{2}\left(1 + \frac{\omega_B^2}{\nu_e^2}\right)V_e^2$$

The source term Q in (4) is defined by

$$Q = 2\nu_e E_k - \beta N_a (\varepsilon_i + \tfrac{3}{2} T_e) \tag{5}$$

The Ohm law (i.e. the reduced electron momentum balance) was used to determine the electric field. V_e – the electron axial velocity field can be expressed in terms of ion velocity and the total current density *eI*:

$V_e = V_i - \frac{I}{n_i}$. The total current density $I = I(t)$ (divided by electron charge e) can be determined from the boundary condition $\int_0^L E dx = U_0$ for the potential drop U_0 to obtain

$$I(t) = \left(\int_0^L \frac{\nu_{eff}}{n_i} dx\right)^{-1} \cdot \left[\frac{e}{m_i} U_0 + \int_0^L \left(\nu_{eff} V_i + \frac{1}{n_i} \frac{\partial}{\partial x} \left(\frac{kT_e}{m_i} n_i \right) \right) dx \right]$$

The system (1)-(4) has four real characteristic velocities

$$V_a \ , \quad V_i - \sqrt{\left(\frac{5}{3}\frac{k}{m_i} T_e\right)} \ , \quad V_i + \sqrt{\left(\frac{5}{3}\frac{k}{m_i} T_e\right)} \ , \quad V_e$$

which are responsible for the propagation velocities of high frequency disturbances of neutrals, ion sound and electron temperature respectively.

If the frequency (hence the wave number) is high enough, RHS of these equations are not influencing the propagation velocity. For lower frequency however, its influence can be manifested very drastically. Especially, the electron energy equation plays an important role in such a case, since its source term (5) is composed of two large terms of opposite signs. This source term introduces a relaxation time τ, related to the characteristic frequency $\omega_c = \tau^{-1}$, which is equal to

$$\omega_c = -\frac{\partial}{\partial T_e} Q \ .$$

For much larger frequencies than ω_c, the source term Q is "too slow" and the temperature is not able to relax, so Q is not influencing the characteristic speeds, of the electron temperature disturbances in particular. In the opposite case however, when $\omega < \omega_c$, the left hand side of Eq. (4) can be neglected, since the temperature can relax, so the electron temperature can be determined from "zero order equation" $Q(T_e, V_i, n_i, N_a, x)=0$ in terms of other dependent variables and the system can be reduced to three partial differential equations for V_i, n_i, N_a.

III. The reduced system

Since we postulate that for low frequencies $Q(T_e, V_i, n_i, N_a, x) = 0$ is a reasonable approximation of (4), therefore in principle the electron temperature may be computed from the equation Q=0 in terms of other variables. It is convenient to do it in two steps. As follows from (5) we may write Q in terms of V_e, N_a, x, having in mind that V_e must be expressed later by V_i and n_i from the definition of total current density: $I(t)= n_i (V_i - V_e)$. Postulating thus that $T_e = T(V_e, N_a, x)$ and eliminating V_e from $I = n_i (V_i - V_e)$ we derive from (3) the new ion momentum equation

$$\frac{\partial V_i}{\partial t} + V_i \frac{\partial V_i}{\partial x} + \frac{1}{m_i}\left(T_e + T_{,V_e} \frac{I}{n_i} \right) \frac{\partial}{\partial x} \ln n_i + T_{,V_e} V_{i,x} + \frac{1}{m_i} \frac{\partial T}{\partial N_a} \frac{\partial N_a}{\partial x} = f \qquad (6)$$

where T_e is not anymore (unknown) dependent variable, $T_{,V_e} = \frac{\partial}{\partial V_e} T_e$ and

$$f = -\nu_{eff}\left(\frac{I}{n_i e} - V_i\right) + \beta N_a (V_a - V_i) + \frac{Q_{,x}}{m_i Q_{,T}}$$

Indeed differentiating relation $Q(T_e, V_e, N_a, x)=0$ we have

$$0 = \frac{\partial}{\partial x} Q(T_e(t,x), V_e(t,x), N_a(t,x), x) = Q,_T T_e,_x + Q,_{Ve} V_e,_x + Q,_{N_a} N_a,_x + Q,_x .$$

Summarizing, the low frequency approximation of (1) –(4) is

$$\frac{\partial N_a}{\partial t} + V_a \frac{\partial N_a}{\partial x} = -\beta N_a n_e$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial (n_i V_i)}{\partial x} = \beta N_a n_e$$

$$\frac{\partial V_i}{\partial t} + V_i \frac{\partial V_i}{\partial x} + \frac{1}{m_i}\left(T_e + T_{e,V_e} \frac{I}{n_i} \right) \frac{\partial}{\partial x} \ln n_i + \frac{1}{m_i} T,_{V_e} V_i,_x + \frac{1}{m_i} \frac{\partial T_e}{\partial N_a} \frac{\partial N_a}{\partial x} = f$$

and

$$Q = 2\nu_e E_k - \beta N_a (\varepsilon_i + \tfrac{3}{2} T_e) = 0$$

This system has obviously different characteristics than Eqs (1-4). They are

$$\lambda_0 = V_a , \quad \lambda_{1,2} = V_i + \frac{1}{2m_i} T_{e,V_e} \pm \sqrt{\frac{1}{m_i}\theta + (\frac{1}{2m_i} T_e,_{V_e})^2} \qquad \text{where } \theta = \left(T_e - T_{e,V_e} \frac{I}{n_i} \right)$$

To estimate the values of interesting for us quantities we have used some results of numerical simulation as presented in [2,3] and the following approximation $\beta = 3.19 \cdot 10^{-13} \frac{\varepsilon - \varepsilon_0}{\varepsilon + \varepsilon_1} \frac{s}{m^3}$

where $\varepsilon = \frac{3}{2} T_e + E_k$, $\varepsilon_0 = 10 eV$, $\varepsilon_1 = 30 eV$. Then we compute

$$T,_{V e} = -\frac{Q,_{E_k}}{Q,_{Te}} E_k,_{V e} = \frac{2}{3} \frac{\frac{\beta,_\varepsilon}{\beta} E_k - 1}{\frac{2}{3}\frac{\nu_{e,T_e}}{\nu_e} - \frac{\beta,_\varepsilon}{\beta} E_k - \frac{1}{2}\beta \frac{N_a}{\nu_e}} \frac{2E_k}{V_e} .$$

Using the identity $\frac{I}{n_i} = v_e \frac{I}{I_e}$, where $I_e = -n_e V_e$ represents the electron current density, we arrive at

$$T,_{V_e} \frac{I}{n_i} = T,_{V_e} V_e \frac{I}{I_e} = -\frac{4}{3} \frac{1 - \frac{\beta,_\varepsilon}{\beta} E_k}{\frac{\beta,_\varepsilon}{\beta} E_k + \frac{1}{2}\frac{\beta N_a}{\nu_e} - \frac{2}{3}\frac{\nu_{e,T}}{\nu_e} E_k} \frac{I}{I_e} E_k$$

To illustrate, that the characteristic speeds may become complex, we take the parameter values from the numerical simulations in [3] at the sonic line, where T_e = 12 eV, E_k = 3 eV, then $\frac{\beta,_\varepsilon}{\beta} E_k = 0.21$, $\frac{I}{I_e} \approx \frac{3}{2}$ noticing that the second and third terms in the denominator of last formulae can be neglected we estimate $\theta \approx$ - 10 eV, So it is negative! On the other hand the term $(\frac{1}{2m_i} T_e,_{V_e})^2$ in the expression for the characteristic velocities is small as compared to $\frac{1}{m_i}\theta$. This indeed shows that for some values of discharge parameters the characteristics of the new system of equations may become complex valued in a certain region of the flow. This however, implies immediately the instability of propagating disturbances. The typical value for the critical frequency in case of SPT-100 is $\omega_c = 10^7$. As follows from analysis made in Sec II we should expect instabilities for frequencies $\omega \ll \omega_c$, say lower than 10^6.

IV Conclusion

Our analysis suggests that the energy equation may significantly contribute to the generation of instabilities in the range of frequencies, usually attributed to so -called transit time instability. In [2], this instability was studied on the basis of isothermal model of plasma discharge. It has been shown that the ion sound wave moving in the same direction as the flow is unstable in the supersonic domain. Our present work shows that the specific properties of the energy balance in Hall thruster plasma can considerably influence these results enhancing the instability.

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