Application of a variational MHD principle to the study of plasma thrusters

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Abstract: An ideal, axial symmetric, single-fluid plasma model is considered in order to describe stationary plasma flows in hybrid magneto plasma dynamic (MPD) thrusters. This magnetohydrodynamic (MHD) description leads to a second order differential equation, the generalized Grad Shafranov equation, for the magnetic stream function ψ and to two implicit constraints relating ψ to the plasma density and to the azimuthal velocity. This set of equations, one differential and two algebraic, is then expressed using a variational approach and its solution is obtained in a straightforward manner from the extremum of the appropriate Lagrangian functional. The adopted numerical approach is based on Ritz's method, which has the advantage of producing analytic (though approximate) solutions. In order to apply this model to the acceleration of plasmas, open-boundary geometries are investigated and specific attention is paid to a physical definition of the boundary conditions. Shocks in the plasma flow are then considered and it is shown that the appropriate jump conditions follow implicitly from a natural extension of the Lagrangian variational principle. Two explicit solutions of plasma flows are presented.

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I. Introduction

Spacecraft propulsion outside the Earth's atmosphere is normally achieved by ejecting a propellant fluid. The thrust is the product of the mass flow rate times a characteristic speed of the ejected flow. The amount of propellant, which greatly influences the weight and size of the mission, hence its cost, must be calculated based on an estimated value of the speed of the ejected flow. There are however chemical and thermodynamic constraints that limit the expansion of the burnt propellant beyond some speed levels. Electrically charged fluids, and in particular plasmas, have been considered over the last decades, as they are subject to electromagnetic interactions. These interactions make it possible to go beyond the limits of thermodynamics: that is why it has become important to understand the fundamentals of plasma dynamics in space propulsion.

When the plasma is stationary and axial symmetry can be assumed, the magnetohydrodynamic (MHD) equations can be reformulated by introducing the function ψ defined in terms of the flux of the poloidal magnetic field. This reduces the problem to the solution of a single partial differential equation, named after the work by Grad[1] and Shafranov[2] (GS). From a mathematical point of view the most effective way to deal with this problem is to use the calculus of variations. A variational formulation of hydromagnetic equilibrium conditions, including the velocity field, was extensively treated for the first time by Woltjer[3,4]. The line of action is to bind the energy of the system with a sufficient number of constraints. However the equations that describe these constraints are known only in the axisymmetric case. In the same line of action of GS, and limiting their analysis to the axisymmetric case, Heinemann & Olbert[5] in 1978 derived a single equation describing the equilibrium of ideal MHD flows and obtained a variational principle from which this equation follows (this is a Lagrangian formulation in the form used in the present paper). Going from the static (zero velocity) case to the conditions of steady state flow ($\partial/\partial t = 0, \mathbf{v} \neq 0$), it is found that the system is governed by two coupled equations, one is a generalized GS equation, a partial differential equation, the other is a generalized Bernoulli equation for fluid dynamics, a nonlinear algebraic equation. The same structure is found for uncharged fluids, though in this case it is much simpler because of the absence of the electromagnetic terms[6]. Presently there is great interest in the variational description. Hameiri[7] demonstrated that, following the ideas present in the paper by Woltjer, it is possible to derive a general (non axisymmetric) variational principle for toroidal configurations. In addition, an elegant variational description of MHD stationary flows is found in Goedbloed[8].

The line of action of the present paper is along the lines of these latter works, with the aim of developing a variational Eulerian model capable of describing the motion of conductive fluids in open domains. This model may be considered as quite limited, but, though it handles quasi-isentropic and axisymmetric stationary flows, it opens up interesting views on astrophysical phenomena, fusion experiments and plasma propulsion. The numerical approach is systematically based on Rayleigh-Ritz's method[9,10], which has the additional advantage of including a simple and effective estimate of the solution.

II. Model

In order to increase the propellant velocity and to maintain control of its flux, the interaction between conductive fluids, or plasmas, and electromagnetic fields has been exploited. Although the plasma acceleration processes present many different possibilities with respect to the non-conductive gasdynamics, the use of a nozzle configuration, like the one described here, seems to improve the thrust performance. In this case, both an external coil and the plasma currents produce a magnetic field that is constrained by a conductive surface. This surface, typically shaped as a hyperboloid of revolution, defines the plasma acceleration channel.

A. Governing Equations

Except for the magnetic force term in the momentum conservation law, Euler's equations can be used to describe the steady flow of both an ideal fluid or a hydromagnetic medium as they provide a convenient approximation of the equations obtained by averaging the kinetic plasma equations. The first assumption made in the derivation of this fluid model is to neglect all dissipative phenomena, such as plasma resistivity and viscosity. This is in general admissible for high energy plasmas and yields a great simplification. More arbitrary is the subsequent assumption of an isotropic pressure tensor, since the local magnetic field direction deeply influences the particle motion. In real applications these assumptions are not verified everywhere in the thruster but provide a very important first step in the investigation of the plasma dynamics.

In steady flow condition, the Eulerian time-derivative vanishes, $\partial/\partial t = 0$, and the *mass conservation law* reads

$$\nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}$$

where ρ and $\mathbf{v}$ represent the fluid density and velocity respectively. Considering the ionization process would require an additional source term in the right hand side of Eq. (1). However, since the plasma is assumed to be fully ionized before the inlet of the accelerating channel, Eq. (1) can be accepted. The momentum equation for a fluid element

$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \mathbf{j} \times \mathbf{B}/c, \tag{2}$$

where p represents the isotropic pressure, expresses the fluid particle acceleration due to pressure gradient and, for conductive fluids, due to magnetic forces. Compared with this term, the electrostatic force can be neglected and space charge effects dropped. Thus the plasma quasi-neutrality is assumed to hold. Next, we assume the conservation of the entropy $S(\rho, p)$ of each fluid element

$$\mathbf{v} \cdot \nabla S = 0. \tag{3}$$

In order to describe the motion of a perfectly conductive fluid interacting with a magnetic field, we need to combine the previous equations with Maxwell's equations

$$\nabla \times \mathbf{B} = 4\pi \mathbf{j}/c, \qquad \nabla \times \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0. \tag{4}$$

Since space charge effects are neglected, Poisson's equation is not included. The last equation which determines the electric field is Ohm's law for a perfectly conductive fluid

$$\mathbf{E} + \mathbf{v} \times \mathbf{B}/c = 0. \tag{5}$$

If a plasma is to be confined by a rigid conductive boundary or by a magnetic field, it is most likely that there will be a transition region where the plasma properties adapt themselves to the presence of boundaries or vacuum. This transition may be a sharp boundary or a gradual change in plasma quantities over some finite distance. In both cases it is necessary to match the variables for the solution of the model equations on adjacent sides of the transition region. Therefore the boundary conditions can be expressed in terms of the change of a plasma variable φ across the interface. The unit vector normal to the interface is denoted by $\mathbf{n}$ and the surface current in the boundary is denoted by $\mathbf{j}_s$. If we define $|\varphi| = \varphi_i - \varphi_o$ the difference between the values assumed by φ on the two sides of the transition region, the boundary conditions result:

Plasma-plasma interface:

$$\left|p + B^2/8\pi\right| = 0, \quad \mathbf{n}\cdot|\rho\mathbf{v}| = 0, \quad \mathbf{n}\times|\mathbf{E}| = \mathbf{n}\times|\mathbf{v}\times\mathbf{B}|/c = 0,$$

$$\mathbf{n}\cdot|\mathbf{B}| = 0, \quad \mathbf{n}\times|\mathbf{B}| = 4\pi\mathbf{j}/c_s.$$

Plasma-perfectly conducting wall interface:

$$\mathbf{n} \cdot \mathbf{v} = 0, \quad \mathbf{n} \times \mathbf{E} = 0, \quad \mathbf{n} \cdot \mathbf{B} = 0.$$

B. Generalized Grad-Shafranov equations

In the following we use cylindrical (r, ϕ, z) coordinates and assume all quantities to be independent of the azimuthal angle ϕ. Let us consider a vector field $\mathbf{C}$ that satisfies the divergence-free condition $\nabla \cdot \mathbf{C} = 0$ (in our case the mass flow $\rho\mathbf{v}$ and the magnetic field $\mathbf{B}$). We define a scalar function ψ independent of ϕ such that the components of $\mathbf{C}$ in the *poloidal* ($\phi = const$) plane can be written as

$$\mathbf{C}_p = \nabla\psi \times \nabla\phi. \tag{6}$$

The function ψ is typically named a *stream-function* since it is constant along the streamlines of the vector field to which it belongs. From Faraday's law it follows that $E_\phi = 0$. By substituting this result into Eq. (5) we obtain

$$\mathbf{v}_p = \kappa(r, z)\mathbf{B}_p. \tag{7}$$

The poloidal magnetic field and the poloidal plasma flux can both be expressed in terms of two different stream functions. However from Eq. (7) we obtain an easy way to express one of these functions in terms of the other and, for this reason, we only define one stream function ψ as:

$$B_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad B_z = \frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{8}$$

For plasma flows this is in general a convenient simplification but it leads to some difficulties in the hydrodynamic limit since the coefficient κ becomes singular for $\mathbf{B}_p \to 0$ and the poloidal velocity cannot be expressed in terms of ψ. This difficulty can be avoided by using two different stream functions or by favouring the one related to the mass flow.

Starting from Eqs. (1-5) it is possible to obtain a set of three equations depending on $\{v_\phi, \rho, \psi\}$, a partial differential equation and two algebraic equations. It can be easily shown that five functions of ψ determine this problem. First, excluding Eq. (2) we obtain

$$F(\psi) = 4\pi\rho\kappa, \quad G(\psi) = (v_\phi - \kappa B_\phi)/r, \quad I(\psi) = p/\rho^\gamma. \tag{9}$$

The azimuthal component of Eq. (2) can be rewritten as $\mathbf{B}_p \cdot \nabla(rB_\phi - rFv_\phi) = 0$, which implies:

$$rB_\phi - rFv_\phi = H(\psi). \tag{10}$$

The component of Eq. (2) parallel to B_p gives

$$\int (dp/\rho)|_{\psi=const} + v^2/2 - rv_\phi G = J(\psi), \tag{11}$$

where $J(\psi)$ represents a generalization of the Bernoulli constant, again a stream function: Eq. (11) takes the name of "generalized Bernoulli equation". The poloidal component parallel to $\nabla\psi$ gives the generalized, nonrelativistic, Grad-Shafranov equation for $\psi(r,z)$[11]:

$$\left(1 - \frac{F^2}{4\pi\rho}\right)\triangle^*\psi - F\nabla\left(\frac{F}{4\pi\rho}\right)\cdot\nabla\psi = -4\pi\rho r^2\left(J' + rv_\phi G'\right) - (H + rv_\phi F)(H' + rv_\phi F') + 4\pi r^2 p\left(S'/k_B\right), \tag{12}$$

where $\triangle^*\psi$ is a differential operator defined as

$$\triangle^*\psi = r\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial z}\right).$$

The functional dependencies of $\{F, G, H, I, J\}$ on the stream function ψ must be assigned. These are determined by arbitrary choice or via experimental measurements. With the appropriate boundary condition, the set of two algebraic equations, Eqs. (10) and (11), and the partial differential Eq. (12) can now be solved in the three unknown fields $\{v_\phi, \rho, \psi\}$.

III. Variational Formulation

A solution of Eq. (12) with the implicit conditions stated by the Bernoulli Eq. (11) and the azimuthal momentum conservation (10) is found taking v_ϕ, ρ and ψ such that the Lagrangian of the system is an extremum,

$$L(v_\varphi, \rho, \psi) = \int_\Omega \mathcal{L}\left(\mathbf{x}, v_\varphi, \rho, \psi, \nabla\psi\right) dV, \tag{13}$$

where $\mathcal{L}$ represents the Lagrangian density of the model. In fact the Euler-Lagrange equations associated to the variational problem are the governing equations derived above. For the classical Grad-Shafranov problem ($\mathbf{v} = 0$) the Lagrangian density depends on ψ only:

$$\mathcal{L}\left(\mathbf{x}, \psi, \nabla\psi\right) = \frac{1}{8\pi}\left(\frac{\nabla\psi}{r}\right)^2 + \frac{1}{8\pi}\left(\frac{Q}{r}\right)^2 + P, \tag{14}$$

where $Q = rB_\phi$ and the plasma pressure P are two functions of ψ. Eq. (14) can be rewritten as

$$\mathcal{L} = -\frac{B_P^2}{4\pi} + \frac{B^2}{8\pi} + p. \tag{15}$$

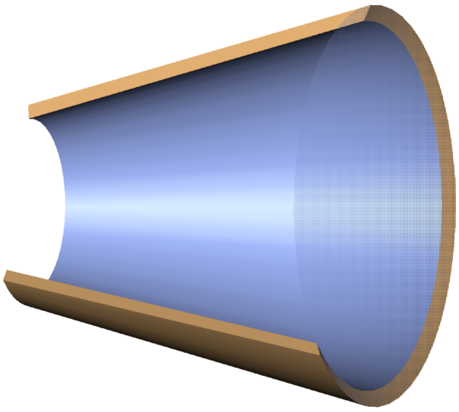

Figure 1. A simple representation of a conic nozzle

The form of $\mathcal{L}$ for the generalized problem is

$$\mathcal{L}\left(\mathbf{x}, v_\varphi, \rho, \psi, \nabla\psi\right) = \left(\frac{F^2}{4\pi\rho} - 1\right)\frac{1}{8\pi}\left(\frac{\nabla\psi}{r}\right)^2 + \frac{1}{8\pi}\left(\frac{H + rv_\phi F}{r}\right)^2 - \frac{1}{2}\rho v_\phi^2 + \rho\left(J + rv_\phi G\right) - \frac{1}{\gamma - 1}\rho^\gamma I, \quad (16)$$

where $\{F, G, H, I, J\}$ are the stream-functions described in the previous section. Eq. (16), is equivalent to

$$\mathcal{L} = -\frac{B_P^2}{4\pi} + \frac{B^2}{8\pi} + p + \rho v_P^2. \quad (17)$$

IV. Boundary Conditions

A crucial question is the study of open boundary configurations. Our basic aim is to obtain a simple representation of the stationary state of an MHD flow through an accelerating nozzle. Since we have to deal with open configurations, we divide the boundaries into two sets: the inlet and outlet surfaces, where the fluid crosses the boundary, and the nozzle walls and the symmetry axis which are assumed to be stream surfaces. This distinction is essential for the prescription of the appropriate boundary conditions. A simple representation of this kind of geometry is the conic nozzle illustrated in Fig. (1) where the inlet and outlet surfaces are taken to be spherical surfaces.

This example is also a test case for the procedure.

In the study of the variational formulation we may consider fixed boundary conditions where the values of the unknown function ψ is assigned on $\partial\Omega$. In another classical type of boundary conditions, relevant to the problem we are investigating, we seek an extremum of the Lagrangian when the values assumed by ψ on a portion $\partial_1\Omega$ of the whole boundary $\partial\Omega$ are not specified. It can be shown that, due to the Lagrangian extremum condition, a specific boundary condition must be satisfied on $\partial_1\Omega$ which yields the following equation

$$\frac{\partial\mathcal{L}\left(\mathbf{x}, v_\varphi, \rho, \psi, \nabla\psi\right)}{\partial\left(\partial\psi/\partial r\right)}\frac{dz}{ds} - \frac{\partial\mathcal{L}\left(\mathbf{x}, v_\varphi, \rho, \psi, \nabla\psi\right)}{\partial\left(\partial\psi/\partial z\right)}\frac{dr}{ds} = 0, \quad (18)$$

where s is the arclength along the boundary. This boundary condition is called the *natural boundary condition* of the problem.

Exploiting the definition of the stream function ψ, we assume the Dirichlet condition on the nozzle wall and on the symmetry axis: $\psi|_{axis} = \psi_0$, $\psi|_{wall} = \psi_1$, whereas on the open portions of the boundary we assume the natural conditions to hold. Since the stream function ψ is defined except for an additive constant, it is convenient to set $\psi_0 = 0$, so that ψ represents the net flux enclosed within the axisymmetric surface that it labels. By substituting the Lagrangian density (16) into Eq. (18), we obtain the equivalent expression of the natural condition for the MHD problem:

$$\frac{1}{r^2}\left(\frac{F^2}{4\pi\rho} - 1\right)\nabla\psi \cdot n = 0.$$

In the open parts of the boundary, where we assume the natural condition to hold, this equation yields

$$\nabla\psi \cdot \mathbf{n}|_{i,o} = 0 \rightarrow \left.\frac{\partial\psi}{\partial n}\right|_{i,o} = 0, \tag{19}$$

where the indexes i and o distinguish respectively the inlet and outlet boundaries. By assigning at the inlet surface B_i as a function of the arclength s we can first express s as a function of ψ and then, assigning ρ_i, p_i, V_i, V_ϕ and B_ϕ at the inlet surface as functions of s, we obtain the explicit expressions of F, G, H, I and J in terms of stream function ψ.

V. Discontinuous Solutions

The variational formulation allows us to find solutions within a more general class of functions. This yields a simple method for including *shocks* in our analysis. It is well known that the hydrodynamic equations are locally elliptic or hyperbolic depending on the ratio between the velocity and the speed of sound (Mach number).

We can see that in some cases, for the same mass flow rate, two different solutions can be found, one with a subsonic flow and the other with a supersonic flow. If we relax the hypothesis of isentropic flow and consider that somewhere in the domain the entropy function changes from S to $S + \Delta S$, it is possible to obtain a new solution with both subsonic and supersonic regimes. This solution and the position of the entropy transition surface depend on the chosen on the value of the entropy jump ΔS. For physically meaningful cases ($\Delta S > 0$), the transition is from a supersonic flow to a subsonic flow and is called a "*shock*" because the fluid properties change discontinuously through it.

An important feature of the variational principle (13) is that the solution of the magnetohydrodynamic flow with *shocks* can be carried out implicitly assuming that the derivatives of the stream function ψ and the density ρ are piecewise continuous.

Let us assume that a discontinuity surface λ (a curve in the poloidal plane) exists which separates the domain Ω into two subdomains: Ω_1 and Ω_2. Thus, given a function $\varphi(x, y)$, we define as $|\varphi| = \varphi_{\lambda^2} - \varphi_{\lambda^1}$ the difference between the two limits $\varphi_{\lambda^2} = \lim_{x\to\lambda}\varphi$, $x \in \Omega_2$, $\quad \varphi_{\lambda^1} = \lim_{x\to\lambda}\varphi$, $x \in \Omega_1$ at a generic point of λ. In a reference frame locally aligned with the discontinuity surface, the velocity and the magnetic field can be written as $\mathbf{V} = v_n\mathbf{n} + v_t\mathbf{t} + v_\phi\phi$, $\quad \mathbf{B} = B_n\mathbf{n} + B_t\mathbf{t} + B_\phi\phi$, where $\mathbf{n}$ and $\mathbf{t}$ represent respectively the normal and the tangential unit vector to λ in the poloidal plane. We determine the jump conditions across the discontinuity by writing Eqs. (1-5) in integral form and by considering an infinitesimal cylindrical volume element with its bases parallel to the shock surface. The mass conservation Eq. (1) yields

$$|\rho v_n| = 0. \tag{20}$$

This same considerations hold for Eqs. (2-5) where, instead of Eq. (3), we use the energy conservation equation

$$\int (\rho\mathbf{V}\cdot\mathbf{n})\left(e + \frac{V^2}{2}\right) dS + \int p\mathbf{n}\cdot\mathbf{V}dS + \int \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\cdot\mathbf{n}dS = 0.$$

In this way we obtain a number of jump conditions equal to the number of conservations.

The same constraints can be found through the extremization of the Lagrangian functional (14) on the set of functions $\mathcal{U} \equiv \{\rho$ *piecewise continuous*, $\psi \in C^0$, $\nabla\psi$ *piecewise continuous*$\}$. The continuity of the stream function ψ and the continuity of the four functions F, G, H and J that depend on ψ give five jump condition. By allowing the plasma density ρ and of the gradient of ψ in the direction normal to the shock to be discontinuous, two additional conditions follow from the variational formulation (details on this derivation can be found in Smirnov[12]).

$$\left|\frac{1}{r^2}\left(\frac{F^2}{4\pi\rho} - 1\right)\nabla\psi\right| \cdot \mathbf{n} = 0, \tag{21}$$

$$|\mathcal{L}| = \left[\frac{F^2}{4\pi\rho} - 1\right]_1 \left[\left.\frac{1}{r^2}\frac{\partial\psi}{\partial z}\right|_1 \left|\frac{\partial\psi}{\partial z}\right| + \left.\frac{1}{r^2}\frac{\partial\psi}{\partial r}\right|_1 \left|\frac{\partial\psi}{\partial r}\right|\right]. \tag{22}$$

From a numerical point of view it is thus possible to model the magnetohydrodynamic flows with shock surfaces within the variational theory by including the position of the shock among the unknowns and by assigning the value of the density jump.

VI. Numerical Procedure

The variational approach allows us to adopt a simple approximation method, the Ritz method, in order to obtain a numerical solution. First, we scale the Lagrangian functional i.e., we cast the model equations in dimensionless form. The extremum is confined inside a finite-dimensional function subspace and the solution is obtained through a system of nonlinear equations. This system is then solved by using the Newton-Raphson algorithm.

A. Dimensionless variables

By scaling all variables with 'typical' values for the configuration of interest, we obtain a dimensionless expression of the Lagrangian density (14)

$$\mathcal{L} = \left(M_1^2 F^2/\rho - 1\right)\left(\nabla\psi/r\right)^2 + (H + M_1 M_2 r v_\phi F)^2/r^2 - M_2^2 \rho v_\phi^2 \tag{23}$$

$$+ M_1^2 \rho J + M_1 M_2 \rho r v_\phi G - 2(M_1^2/M_0)^2 \rho^\gamma I/[\gamma(\gamma-1)],$$

where we have introduced the three dimensionless numbers that characterize the solutions: the Mach number $M_0 \equiv V_0/\sqrt{\gamma p_0/\rho_0}$, the Mach-Alfvèn number $M_1 \equiv \sqrt{F_0^2/4\pi\rho_0}$ and the Mach-Alfvèn azimuthal number $M_2 \equiv \sqrt{4\pi\rho_0 V_{0\phi}^2/B_0^2}$ and the subscript zero denotes normalization quantities.

B. The Ritz Method

We assume that the approximation functions belong to a finite dimensional subspace of the solution space. and consider the expansion of ψ, ρ and v_ϕ as sums of base-functions

$$\psi(s,t) = \sum_{i.j=0}^{n_i,n_j} \psi_{ij}\, F_i^s(s)\, F_j^t(t), \qquad \rho(s,t) = \sum_{i.j=0}^{m_i,m_j} \rho_{ij}\, G_i^s(s)\, G_j^t(t), \tag{24}$$

$$v_\phi(s,t) = \sum_{i.j=0}^{l_i,l_j} v_{\phi ij} H_i^s(s)\, H_j^t(t),$$

where $F^{s,t}, G^{s,t}$ and $H^{s,t}$ are three families of base-functions. Inserting Eq. (24) into Eq. (23) we obtain

$$L = \int_\Omega \mathcal{L}\left(\mathbf{x}, v_\phi(v_{\phi ij}), \rho(\rho_{ij}), \psi(\psi_{ij}), \nabla\psi(\psi_{ij})\right) dV, \tag{25}$$

and the extremization process can be developed differentiating Eq. (25) with respect to the three sets of coefficients $\{\psi_{ij},\ \rho_{ij},\ v_{\phi ij}\}$

$$\frac{\partial L}{\partial \psi_{ij}} = \int_\Omega \frac{\partial \mathcal{L}}{\partial \psi_{ij}} dV \equiv P_{ij}, \quad \frac{\partial L}{\partial \rho_{ij}} = \int_\Omega \frac{\partial \mathcal{L}}{\partial \rho_{ij}} dV \equiv R_{ij}, \quad \frac{\partial L}{\partial v_{\phi ij}} = \int_\Omega \frac{\partial \mathcal{L}}{\partial v_{\phi ij}} dV \equiv V_{ij}.$$

The approximation coefficients are a solution of the nonlinear equation system

$$P_{ij} = R_{ij} = V_{ij} = 0, \tag{26}$$

for all values of the summation indices i, j. We notice that in general it is not possible to state that the solution is a minimum (or a maximum) of the Lagrangian of the system. Thus the convergence theorem for the Ritz method applies only if some additional assumptions hold. Since each equation of the nonlinear system (26) is a smooth function of the unknown coefficients and can be easily differentiated with respect to these coefficients, we can solve it by exploiting Newton-Raphson's algorithm.

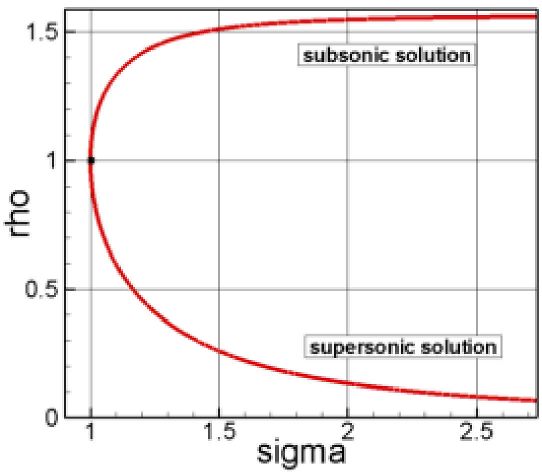

Figure 2. Subsonic and supersonic solutions for the density ρ as a function of the spherical radius σ for inlet sonic conditions. Both curves are plotted in dimensionless units.

C. Test Case

The above procedure has been tested on some particular solutions of the hydrodynamic and MHD models that can be expressed in analytic form or easily interpolated. More importantly a solution of the hydrodynamic model with an entropy discontinuity has been determined and the variational procedure described above for shock capture has be tested.

As an example here we refer to the very simple case of constant inlet conditions ($\mathbf{B}\cdot\mathbf{n}|_i = B_0$, $\mathbf{v}\cdot\mathbf{n}|_i = V_0, \rho|_i = \rho_0$ and $p|_i = p_0$), zero azimuthal velocity ($v_\phi = 0$) and zero azimuthal magnetic field ($B_\phi = 0$), in which case it is possible to determine an analytic solution of the conic nozzle problem. First we deduce the five stream functions $[F, G, H, I, J]$ from the inlet conditions: the ratio between the mass flow rate and the magnetic field in the poloidal plane gives $F(\psi) = 4\pi\rho_0 V_0/B_0 = F_0$, the zero azimuthal velocity and azimuthal magnetic field imply $G(\psi) = 0$ and $H(\psi) = 0$, the Bernoulli equation written at the inlet surface leads to $J(\psi) = V_0^2/2 + [\gamma/(\gamma-1)]p_0/\rho_0 = J_0$, and the flux entropy is uniform and the related function is $I(\psi) = p_0/\rho_0^\gamma$. By simplifying the terms on the right hand side of Eq. (12), we obtain in spherical coordinates, $\sigma = (r^2+z^2)^{1/2}$ and θ,

$$\frac{\partial}{\partial\sigma}\left[\frac{1}{\rho}\frac{\partial\psi}{\partial\sigma}\right] + \frac{\sin\theta}{\sigma^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{1}{\rho}\frac{\partial\psi}{\partial\theta}\right] = 0,$$

where the density ρ is given by the algebraic Bernoulli Eq. (11)

$$J_0 = \frac{1}{2}\left(\frac{F_0}{\sigma\sin\theta}\frac{\nabla\psi}{4\pi\rho}\right)^2 + \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0^\gamma}\rho^{\gamma-1}. \tag{27}$$

Considering a stream function of the form $\psi = \sigma_0^2 B_0[1-\cos\theta]$ and inserting it into Eq. (27), we find that the density depends on the spherical radius only, $\rho = \rho(\sigma)$, and obeys the polynomial equation

$$(\rho/\rho_0)^{\gamma+1} - (K+1)(\rho/\rho_0)^2 + K(\sigma/\sigma_0)^{-4} = 0, \tag{28}$$

with K defined by $K = V_0^2(\gamma - 1/\gamma)(\rho_0/2p_0) = (\gamma-1)M_0^2/2$.

Eq. (28) can be interpolated numerically and the result can be compared with that obtained from the variational algorithm. The input parameters have been chosen according to the standard operational value of the Hybrid Plasma Thruster described in Ref. 13. The flow velocity in the inlet region is taken equal to the speed of sound and the flow after it is supposed to be supersonic. Both subsonic and supersonic solutions of Eq. (28) are shown in Fig. (2).

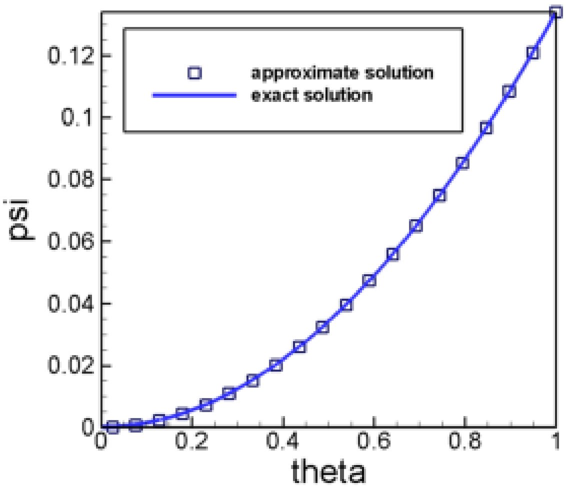

Figure 3. Exact and approximate stream function solutions of the test case in Sec. (VI-C), plotted in dimensionless unit.

In Fig. (2) the upper branch shows a typical subsonic flow ($M < 1$) while the lower (decreasing) branch is characteristic of supersonic ($M > 1$) regimes.

The comparison between the exact solution and our approximated results is good, as can be seen in Figs. (3)-(4)

VII. An example of thruster operation

In this section we explore a configuration closer to the condition of thruster operation. As in Sec. (VI-C) we consider a conical nozzle. The function $G(\psi)$, that defines the electric field in the plasma, and the function $H(\psi)$ are both related the azimuthal inlet velocity and azimuthal magnetic field. We choose these two functions in the inlet region so as to give

$$G(\psi) = G_0 \qquad H(\psi) = 0,$$

and leave all other quantities unchanged. Thus, the values of $F(\psi)$ and $I(\psi)$ remain the same as in the previous case in Sec. (VI-C). For the azimuthal velocity we obtain $v_\phi|_i = V_{\phi 0} \sin\theta$ and $G_0 = (V_{\phi 0}/\sigma_0)\left(1 - M_1^2\right)$. The generalized Bernoulli equation becomes

$$J(\psi) = [V_0^2 + V_{\phi 0}^2 \sin^2\theta]/2 + [\gamma/(\gamma-1)]p_0/\rho_0 - \sigma_0 V_{\phi 0} G_0 \sin^2\theta, \tag{29}$$

where $\theta|_i = \theta(\psi)$ can be obtained by integrating $\mathbf{B}\cdot\mathbf{n}|_i$ which gives $\psi|_i = \sigma_0^2 B_0 [1 - \cos(\theta)]$. Thus Eq. (29) can be rewritten in the quadratic form

$$J(\psi) = J_0 + J_1(\psi/\psi_0) + J_2(\psi/\psi_0)^2$$

where $\psi_0 = \sigma_0^2 B_0$, $J_0 = (V_0^2/2)\left[1 + 1/[(\gamma-1)M_0^2]\right]$, $J_1 = V_{\phi 0}^2\left(2M_1^2 - 1\right)$, and $J_2 = -J_1/2$. The numerical solution of this case obtained with the variational method is illustrated in Figs. (5)-(7).

VIII. Conclusion

The theoretical model of an axially symmetric plasma flow has been discussed in its fundamental features: the governing equations have been presented, a variational principle has been formulated and boundary and jump conditions have been derived. A numerical approach based on a variational formulation has been described in detail.

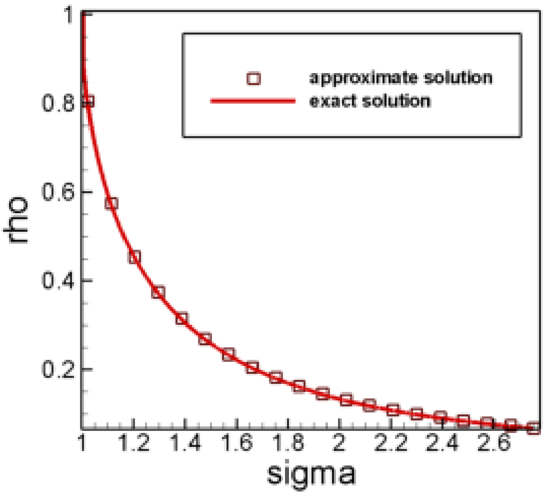

Figure 4. Exact and approximate density solutions of the test case in Sec. (VI-C), plotted in dimensionless unit.

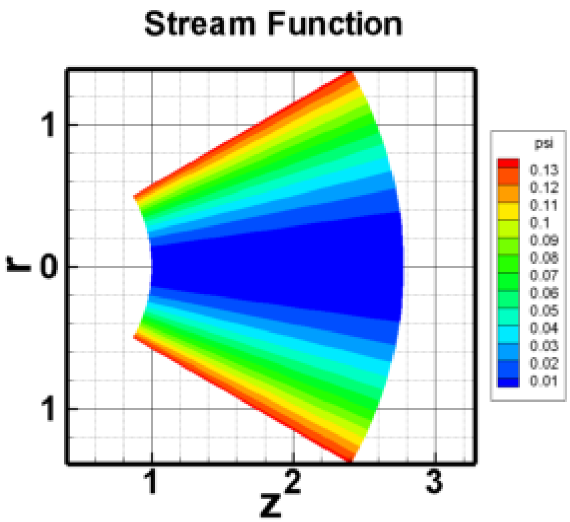

Figure 5. Contour plot of the stream function ψ for the example of thruster operation described in Sec. (VII).

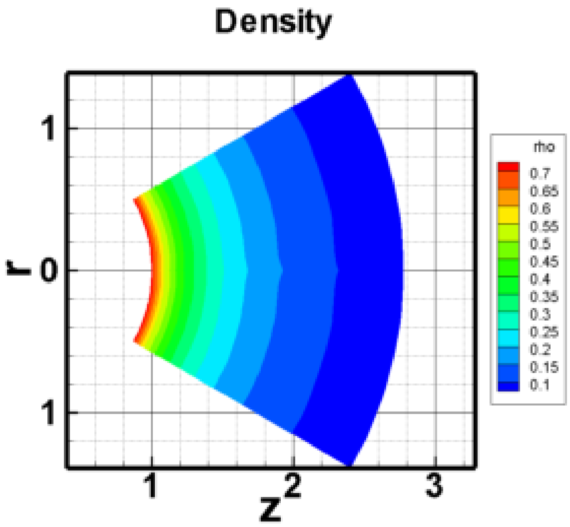

Figure 6. Contour plot of the density function ρ for the example of thruster operation described in Sec. (VII).

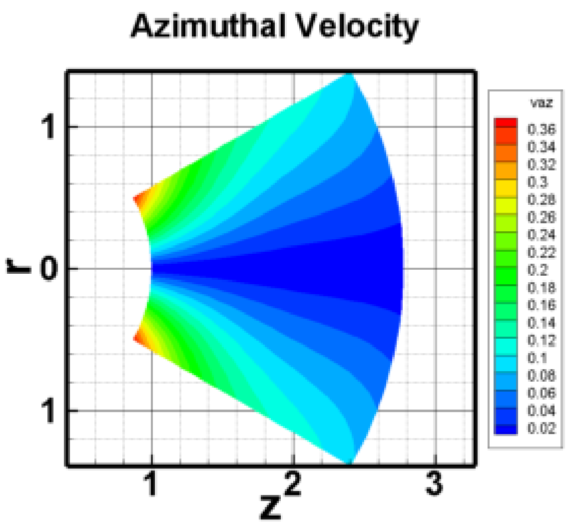

Figure 7. Contour plot of the azimuthal velocity function v_ϕ for the example of thruster operation described in Sec. (VII).

This variational method has been implemented in a numerical code by using the C++ language and solutions of open boundaries plasma flows have been obtained. The relative simplicity of the code and the facts that it can include open boundaries in a natural fashion will allow us to investigate a wide range of different plasma flows beyond the simple cases discussed in the present article.

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