

A new breath for the breathing mode

IEPC-2009-070

*Presented at the 31st International Electric Propulsion Conference,
University of Michigan, Ann Arbor, Michigan, USA
September 20–24, 2009*

Serge Barral*

Institute of Plasma Physics and Laser Microfusion, 01-497 Warsaw, Poland

and

Zbigniew Peradzyński†

Warsaw University, 02-097 Warsaw, Poland

Recent advances in the one-dimensional non-linear theory of the breathing mode in Hall thrusters are reviewed, with an emphasis on the physical interpretation of the stability criterion. Pursuing the apparent analogy between the one-dimensional model and predator-prey dynamics, a simple predator-prey model is derived which, unlike the marginally stable Lotka-Volterra model, possesses distinct regions of stability and instability. A correlation of this improved predator-prey model with the one-dimensional theory suggests that instability results from a delayed feedback between ionization and avalanche electron multiplication, incurred by the transit of neutrals when the ionization region begins upstream of the avalanche region.

Nomenclature

| | |
|------------|---|
| f_e, f_i | Electron and ion distribution functions |
| I | Discharge current |
| ℓ | Length of the discharge column |
| L | Length of the predator-prey dynamics region |
| n | Plasma density |
| N | Density of neutrals |
| \bar{N} | Density of neutrals in the steady state |
| Q_0 | Flux density of neutrals in the injection plane |
| v | Velocity of expelled ions |
| V | Velocity of neutrals |
| β | Electron multiplication rate |
| γ | Effective local ionization rate |
| Ψ | Local propensity to electron avalanche multiplication |
| τ | Transit time of neutrals in the pre-ionization region |

I. Introduction

BREATHING discharge current oscillations constitute one of the most characteristic phenomena of Hall accelerator discharges, and one of the first to be successfully recovered in numerical simulations.^{1,2} Yet, the physics at play

*Assistant Professor, Division of Magnetised Plasma, sbarral@ifpilm.waw.pl.

†Professor, Institute of Applied Mathematics and Mechanics, zperadz@mimuw.edu.pl.

has for long remained elusive. The interpretation of this mode as a predator-prey ionization cycle¹ has *de facto* gained wider acceptance than other semi-empirical theories,^{3,4} but this acceptance has remained a matter of trust and intuition in the absence of a compelling validation against simulations or experiments.

A comprehensive mathematical model of the breathing mode based on first principles has been recently derived.^{5,6} The model obtained for linear modes⁵ has been shown to reduce in substance to a linearized predator-prey model. Non-linear modes have been in turn identified to relaxation oscillations.⁶ The non-linear modes predicted by a simple model have been unambiguously shown to recover the mode frequency, the back-and-forth motion of the ionization front and the shape of discharge current pulses predicted by numerical simulations.⁶

In this work, the reduction of the one-dimensional model of ionization oscillations to a volume-averaged model is investigated, with the intent to improve and validate the predator-prey interpretation.¹

II. Summary of the one-dimensional theory of breathing oscillations

A. Model

The formal derivation of the non-linear one-dimensional theory of breathing oscillations is given elsewhere in details,⁶ and we shall limit ourselves to a concise summary of its governing principles. Assuming that the discharge potential applied between the anode ($x = 0$) and the cathode ($x = \ell$) is constant,

$$\int_0^\ell E dx = \text{const}, \quad (1)$$

one can use the Boltzmann equations for ions and electrons to show⁶ that the secular behavior (i.e. the long term trends) of the discharge current I and of the distribution functions f_i and f_e obeys,

$$\frac{dI}{dt} = I \mathcal{J}(N), \quad (2)$$

$$f_e = I \mathcal{F}_e(N), \quad f_i = I \mathcal{F}_i(N), \quad (3)$$

where \mathcal{J} , \mathcal{F}_e and \mathcal{F}_i have a global functional dependence on the density of neutrals, meaning that $I^{-1}dI/dt$ and the local values of $I^{-1}f_e$ and $I^{-1}f_i$ depend at any moment on the whole profile $N(x)$ within the discharge column.

The interpretation of Eq (2) is the following: when the density of neutrals N is “too large”, ionization makes the discharge current to increase exponentially in time, at a rate that depends on the excess of neutrals. This avalanche instability follows from the excess production of ions, which in turn draw more electrons into the discharge to maintain quasineutrality. Owing to the constant discharge potential, these new electrons have enough energy to further enhance ionization, leading to avalanche ionization. Conversely, the discharge current decreases continuously when N is too small. Since the dynamics of charged species is much faster than that of neutrals, the shape of the distribution functions of charged species remains in a quasi-equilibrium with the profile of N , notwithstanding the variation of the plasma density in proportion to I predicted by Eqs (3).

Any increase of discharge current and plasma density subsequent to an excess of neutrals results, however, in a depletion of neutrals that eventually prevents the continuation of the avalanche process. The current then decreases until the repletion of neutrals is sufficient to trigger again avalanche ionization. This is the essence of the mechanism of breathing oscillations, which in its main lines is consistent with the qualitative descriptions of Boeuf *et al.*² and Fife *et al.*¹

Skipping the details,⁶ Eqs (2, 3) eventually lead to a simple model of ionization oscillations,

$$\frac{dI}{dt} = I \int_0^\ell \Psi \ln \frac{N}{\bar{N}} dx, \quad (4a)$$

$$\frac{\partial N}{\partial t} + V \frac{\partial N}{\partial x} = -\gamma N I, \quad (4b)$$

$$N(x=0) = \frac{Q_0}{V}. \quad (4c)$$

where Eq. (4a) is obtained by linearization of functional \mathcal{J} in Eq. (2), and where Eqs (4b, 4c) describe the advective-reactive transport of neutrals, introduced at the anode with a constant flow rate. Function $\Psi(x)$ can be thought of as the

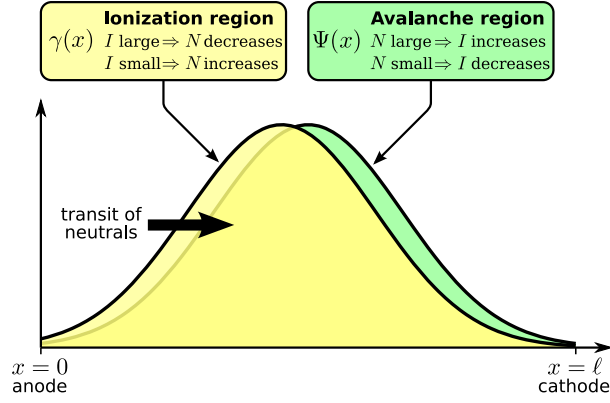


Figure 1. Idealized shapes of the effective ionization rate (γ) and of the propensity to electron avalanche multiplication (Ψ) inferred from numerical simulations⁵.

sensitivity of electron avalanche multiplication to a local perturbation of the density of neutrals, while function $\gamma(x)$ can be identified to an effective ionization rate that integrates the spatial variations of the electron density. $\bar{N}(x)$ is the steady-state density profile of neutrals.

B. Typical behavior

The one-dimensional model recovers the main characteristic features of the breathing mode highlighted by Boeuf *et al.*² and others,^{7–10} such as the slow progression and fast recession of the ionization front or the shape and large magnitude of discharge current oscillations. Figure 2 illustrates the predicted discharge evolution with a low-current startup. The assumed shapes of γ and Ψ , shown on Fig. 1, are given by $\gamma = 2 \exp[-16(x/\ell - 0.45)^2]$ and $\Psi = 2 \exp[-16(x/\ell - 0.5)^2]$ and closely fit the actual functions extracted from conventional numerical simulations.⁵ The limit cycles shown by Fig. 1 are in all respects similar to those of typical numerical simulations.^{2,5,8–10}

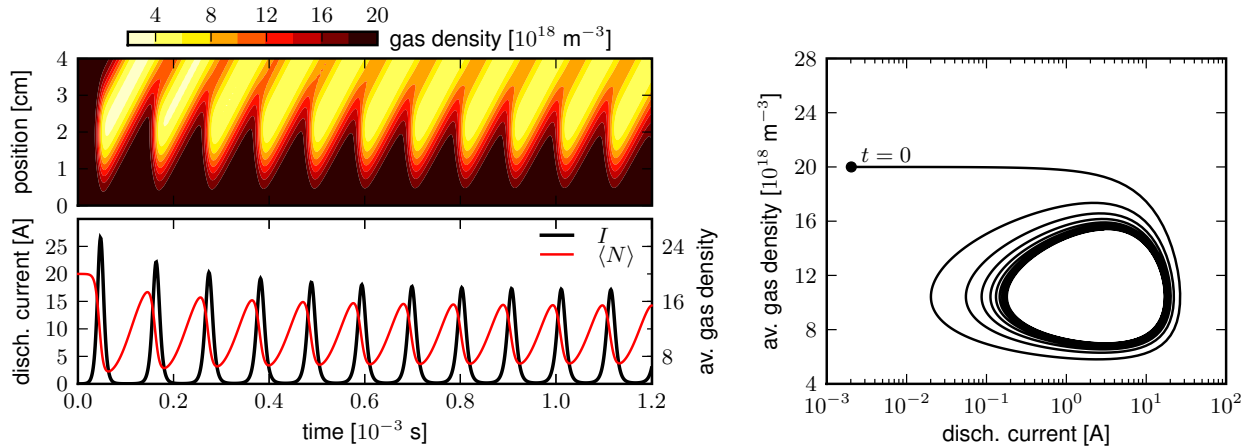


Figure 2. Convergence of the one-dimensional model (4) to a limit cycle with a low-current, gas-filled startup. The phase portrait shows the coupled evolution of the discharge current and of the average gas density within the domain. The parameters are $\ell = 4$ cm, $V = 250$ ms⁻¹, $Q_0 = 5 \times 10^{21}$ m⁻²s⁻¹, $\bar{I} = 4$ A, $\gamma_0 = 4 \times 10^3$ s⁻¹A⁻¹, $\Psi_0 = 2 \times 10^7$ s⁻¹m⁻¹. The assumed profiles $\gamma(x)$ and $\Psi(x)$ are shown on Fig. 1.

C. Stability

An asymptotic study of model (4) when the typical plasma to gas density ratio n/N is very small can be performed to show that a sufficient condition for the existence of ionization oscillations is:

$$\int_0^\ell \Psi(x) \gamma'(x) dx - \gamma(0) [\Psi(\ell) - \Psi(0)] < 0. \quad (5)$$

Interestingly, it can be demonstrated that the exact same condition is necessary and sufficient for the model to be linearly unstable in the small n/N limit. The above relation can be simplified by arguing that γ and Ψ are negligible in the anode and cathode plane, in which case instability is expected when,

$$\int_0^\ell \Psi(x) \gamma'(x) dx < 0. \quad (6)$$

We note that the relation $\int_0^\ell \Psi \gamma' = -\int_0^\ell \Psi' \gamma$ always holds for functions vanishing at both ends, which allows condition $\int_0^\ell \Psi \gamma' < 0$ to be objectively interpreted as “function γ is located before Ψ ”, since it is mutually exclusive with $\int_0^\ell \gamma \Psi' < 0$, *i.e.* with the postulate “function Ψ is located before γ ”.

Summarizing (in a less formal language) we may succinctly interpret this condition as follows:

For small plasma-to-gas density ratios, ionization oscillations occur whenever the ionization zone begins upstream (from the perspective of gas transport) of the electron avalanche region.

This interpretation readily suggests that instability may relate to the time delay incurred by the transit of neutrals from the beginning of the ionization region to the electron avalanche multiplication region, a conjecture which legitimacy shall be assessed in the sequel with a modified predator-prey model.

III. A better predator-prey model

A. Background

The interpretation of breathing oscillations suggested by Fife *et al.*¹ is based on the classical Lotka-Volterra model:

$$\frac{dn}{dt} = \beta N n - \frac{v}{L} n, \quad (7a)$$

$$\frac{dN}{dt} = -\beta N n + \frac{V}{L} N, \quad (7b)$$

where n is understood as the volume-averaged plasma density and N as the volume-averaged density of neutrals. Although this model is mostly conform to the intuition, the birth term for neutrals appears quite unphysical: it implies that neutral enter the ionization region at a rate proportional to N , when in fact neutrals are introduced in the channel at a constant rate. This statement also breaks causality by stating that neutrals “know” their density in the ionization region before they actually enter it. Notwithstanding phenomenological considerations, the Lotka-Volterra equations are unable to determine the stability (or instability) of the system: the amplitude and frequency of oscillations are solely determined by the initial conditions, as illustrated by Fig. 3. This often overlooked property also implies that the frequency of linear modes¹ may not reflect that of non-linear modes, as confirmed by numerical simulations.⁵

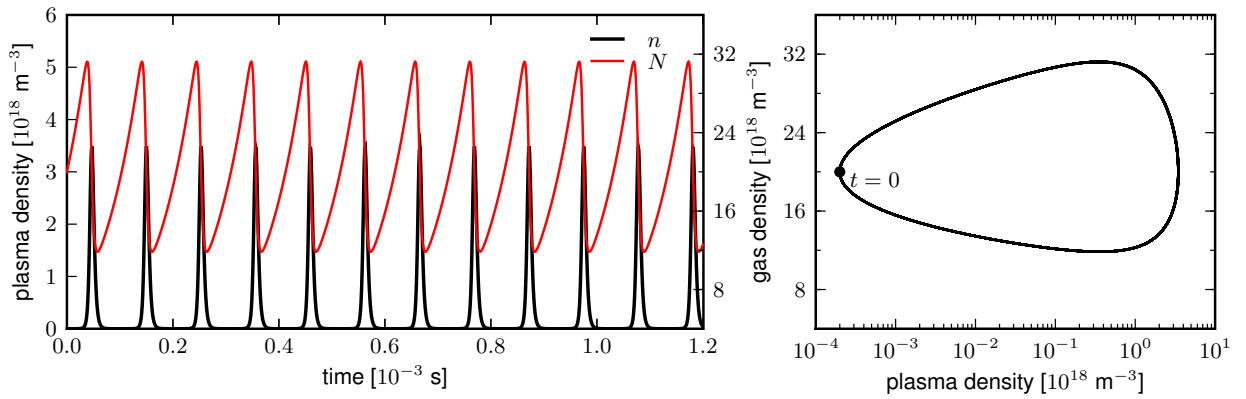


Figure 3. Oscillations predicted by the classical predator-prey model (7) with a low-current, gas-filled startup. The limit-cycle is solely determined by the initial conditions. The parameters are derived from those of Fig. 2, assuming that the length of the ionization region represents approximately half of the discharge column length: $L = \ell/2 = 2 \text{ cm}$, $V = 250 \text{ ms}^{-1}$, $v = Q_0 \beta L/V = 14 \text{ kms}^{-1}$, $\beta \approx \int_0^\ell \Psi/N dx = 3.5 \times 10^{-14} \text{ m}^3 \text{ s}^{-1}$.

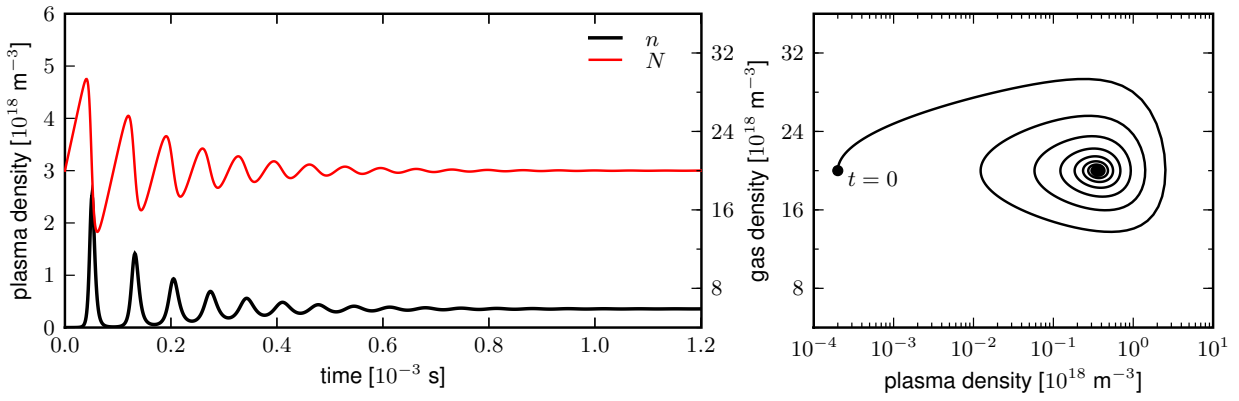


Figure 4. Behavior of the constant gas flow predator-prey model (7a, 8), with a low-current, gas-filled startup. This model is unconditionally stable. The parameters are those of Fig. 3, with $Q_0 = 5 \times 10^{21} \text{ m}^{-2} \text{ s}^{-1}$.

For these reasons, the original Lotka-Volterra equations are seldom used as such in modern ecological dynamics. Ecological models usually include various processes such as *stocking* (the continuous supply of a species), *harvesting* (the continuous removal of a species) and time delays, which ultimately determine the stability of the system. With this perspective, it is tempting to modify Eq. (7b) by representing the constant flow rate of the working gas with constant prey stocking,

$$\frac{dN}{dt} = -\beta Nn + \frac{Q_0}{L}, \quad (8)$$

where the stocking rate is Q_0/L . It can be easily shown, however, that this system is unconditionally stable and thus of little interest to explain ionization oscillations (see Fig. 4).

Recalling the instability criterion given in Sec. C, the lack of unstable solutions in the above predator-prey models is not totally unexpected as these models do not contain any notion of location regarding the ionization and avalanche regions.

B. Formulation

Integrating locational information in a zero-dimensional model while retaining the essence of the ionization mechanism is not as straightforward a task as it may seem. An obvious way to account for a possible shift between the ionization and avalanche regions would consist in adding a delay τ before the population N influences electron avalanche multiplication, changing $N(t)$ for $N(t - \tau)$ on the right hand side of Eq. 7a. This indeed provides for unstable modes, but misses the fact that most neutrals are actually already in the avalanche zone when avalanche occurs and provide thus an instantaneous feedback which deeply affects the dynamics of the system. Other obvious formulations in terms of classical time-delayed predator-prey models have been found equally unsuitable to recover the behavior predicted by the one-dimensional model.

Doubtlessly, the reduction from one to zero dimension invariably implies compromises, but it turns out that a fairly realistic behavior can be recovered by distinguishing the population of neutrals in the avalanche region from the population in the *pre-ionization* region, *i.e.* in the part of the ionization region located upstream from the avalanche region (when such exists). The corresponding model is derived formally in Appendix A and reads,

$$\frac{dI}{dt} = \beta I(N - \bar{N}), \quad (9a)$$

$$\frac{dN}{dt} = -\gamma IN + \frac{Q_0}{L} \exp\left[-\gamma \int_{t-\tau}^t Idt\right]. \quad (9b)$$

Noting that \bar{N} , β and γ are constants, the above model can be considered formally identical to the Lotka-Volterra model (7) with $I = (\beta/\gamma)n$, save for the stocking rate on the right hand side of Eq. (9b) which also accounts for pre-ionization. A typical low-current startup of the model is shown in Fig. 5, where the parameters have been chosen to remain in a loose consistence with those of Fig. 2.

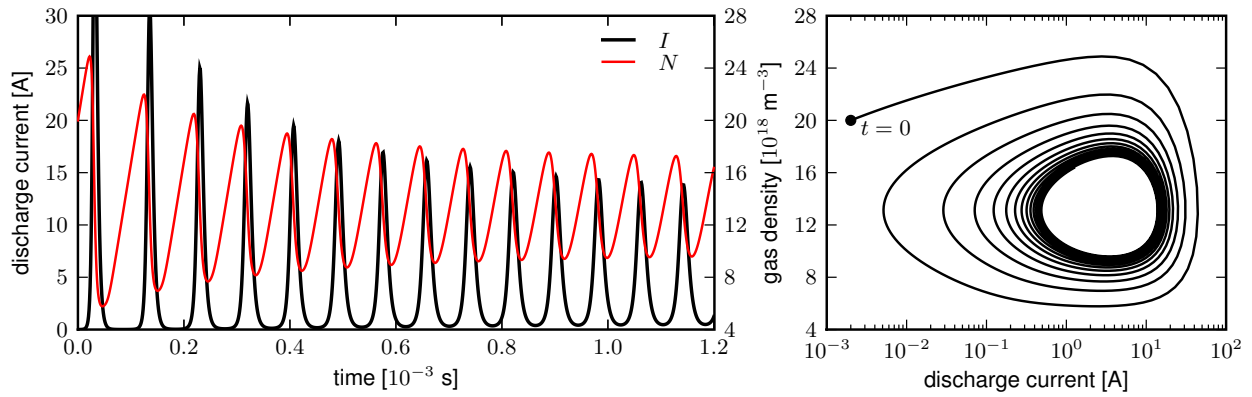


Figure 5. Convergence of the modified predator-prey model (9) to a limit cycle with a low-current, gas-filled startup. The parameters are derived from those of Fig. 2, assuming that the length of the ionization region represents approximately half of the discharge column length: $L = \ell/2 = 2 \text{ cm}$, $Q_0 = 5 \times 10^{21} \text{ m}^{-2} \text{ s}^{-1}$, $\bar{I} = 4 \text{ A}$, $\gamma = 3.5 \times 10^3 \text{ s}^{-1} \text{ A}^{-1}$, $\beta \approx \int_0^L \Psi/\bar{N} dx = 3.5 \times 10^{-14} \text{ m}^3 \text{ s}^{-1}$, $\tau = 2.2 \times 10^{-5} \text{ s}$.

C. Properties

Let us now verify that the main properties of the derived predator-prey model are consistent with those of the one-dimensional model.

Just as in the case of the one-dimensional mode, theoretical calculations can be greatly simplified by making use of the fact that the ratio $\varepsilon \equiv \bar{n}/\bar{N}$ is a small parameter. The non-dimensionalization procedure used in the one-dimensional model⁶ to obtain an explicit formulation in terms of ε can be avoided by noting that β varies in proportion to $1/\varepsilon$ in the case where \bar{I} , Q_0 , γ and τ are fixed. Accordingly, an asymptotic analysis for large β can be performed from which one obtains the frequency and damping rate for linear modes,

$$\text{Re}(\omega) \approx \sqrt{\beta \frac{Q_0}{L} \exp(-\gamma \bar{I} \tau)}, \quad (10)$$

$$\text{Im}(\omega) \approx \frac{\gamma \bar{I}}{2} \cos[\tau \text{Re}(\omega)]. \quad (11)$$

The frequency formula (10) is the same as that of linear Lotka-Volterra modes,¹ save for the pre-ionization correction factor $\sqrt{\exp(-\gamma \bar{I} \tau)}$ which is typically close to 1. The damping rate shows that the model is linearly stable for small τ , but becomes unstable when τ crosses the value $\tau_c = \pi/[2\text{Re}(\omega)]$, as visible on Fig. 6.

Another common point with the one-dimensional model is the fact that the linear frequency scaling in $\sqrt{\beta}$ (or equivalently, in $1/\sqrt{\varepsilon}$) does not reflect the scaling of non-linear modes. As can be seen on Fig. 6, the frequency of non-linear modes rapidly converge as β increases, which is consistent with the fact that there exists an asymptotic frequency at

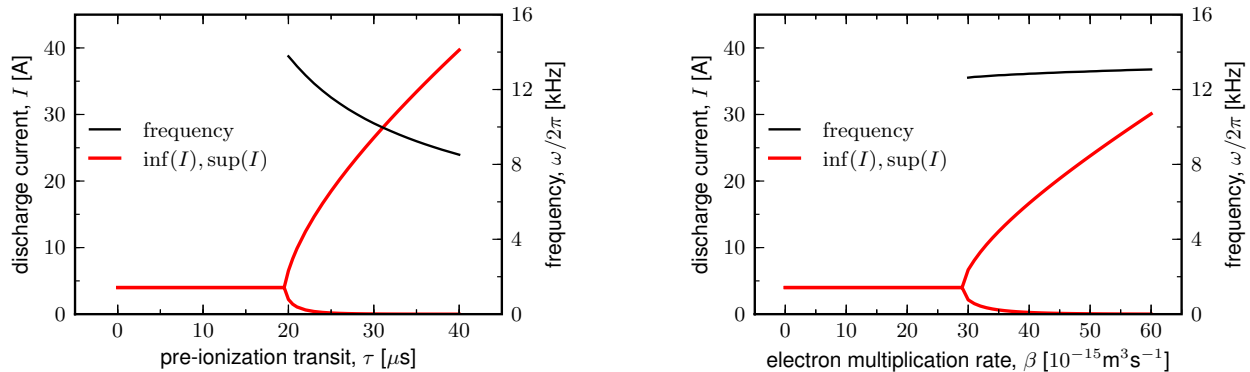


Figure 6. Bifurcation diagrams for the modified predator-prey model (9). Parameters τ and β are varied around the nominal point investigated in Fig. 5.

small ε in the one-dimensional model.⁶ An analytical expression of this asymptotic frequency remains to be found in the case of the predator-prey model, but Fig. 6 strongly suggests that, just like for the one-dimensional model, the frequency of non-linear modes is of the order of magnitude of the steady state gas ionization frequency, $\nu_i \equiv \gamma \bar{I}$.

IV. Conclusion

The basic features of ionization oscillations exhibited by the one-dimensional model^{5,6} are reasonably captured by a modified Lotka-Volterra model where the source term for neutrals accounts for the constant flow rate at the gas feed and its subsequent depletion in a pre-ionization region.

The predator-prey model converges to a limit cycle independent of initial conditions and the limit cycle frequency remains largely independent of the electron multiplication rate, both properties being consistent with the one-dimensional mode. The stability threshold of the predator-prey model is determined by the time-delay incurred by the transit of neutrals through the pre-ionization region, which also appears to coincide with the stability criterion of the one-dimensional model.

A. Formal derivation of the predator-prey model

A simple predator-prey type model can be formally derived from the one-dimensional model (4) under the following assumptions:

1. γ is constant in the whole ionization region and zero elsewhere,
2. Ψ/\bar{N} is constant in the avalanche zone and zero elsewhere,
3. the variations of N around the steady state \bar{N} are small and justify the approximation $\ln(N/\bar{N}) \approx N/\bar{N} - 1$; this hypothesis is anyway implicit in the linearization that leads from Eq. (2) to Eq. (4a),
4. the flux of non-ionized neutrals leaving the avalanche zone is negligible.

Using assumption 1, Eq. (4b) can be integrated along characteristic lines to obtain the value of the flux of neutrals at the exit of the pre-ionization region (i.e. at the entrance of the avalanche region),

$$Q(t) = Q_0 \exp \left[-\gamma \int_{t-\tau}^t I(t^*) dt^* \right], \quad (12)$$

where τ is the transit time of neutrals in the pre-ionization region (from the beginning of the ionization region to the beginning of the avalanche region). Using in turn assumptions 2, one can derive from Eqs (4b, 4a) the relation,

$$\frac{dI}{dt} = \beta I (\langle N \rangle - \langle \bar{N} \rangle), \quad (13)$$

where $\langle N \rangle$ represents the average density in the avalanche region and where $\beta \equiv \int \Psi/\bar{N} dx$. Finally, under assumptions 1 and 4, Eq. (4b) can be averaged over the extent of the avalanche region to obtain,

$$\frac{d\langle N \rangle}{dt} = -\gamma I \langle N \rangle + \frac{Q}{L}, \quad (14)$$

where the flux density Q of neutrals entering the avalanche region is given by Eq. 12.

Acknowledgments

This work has been performed with funding from the French Research Group ‘‘Propulsion par Plasma dans l’Espace’’ (GDR 3161 CNRS/CNES/SNECMA/Universit s).

References

- ¹J. M. Fife, M. Mart nez-S nchez, and J. Szabo. A numerical study of low-frequency discharge oscillations in Hall thrusters. In *Proc. 33rd AIAA Joint Propulsion Conference*, number 97-3051, Seattle, WA, 1997. American Institute of Aeronautics and Astronautics, Washington, D.C.

- ²J.-P. Boeuf and L. Garrigues. Low frequency oscillations in a stationary plasma thruster. *J. Appl. Phys.*, 84:3541, 1998.
- ³N. Yamamoto, K. Komurasaki, and Y. Arakawa. Discharge current oscillation in Hall thrusters. *J. Propul. Power*, 21:870, 2005.
- ⁴S. Chable and F. Rogier. Numerical investigation and modeling of stationary plasma thruster low frequency oscillations. *Phys. Plasmas*, 12, 2005.
- ⁵S. Barral and E. Ahedo. Low-frequency model of breathing oscillations in Hall discharges. *Phys. Rev. E*, 79, 2009.
- ⁶S. Barral and Z. Peradzyński. Ionization oscillations in Hall accelerators. submitted, 2009.
- ⁷J. M. Fife, M. Martínez-Sánchez, and J. Szabo. A numerical study of low-frequency discharge oscillations in Hall thrusters. In *Proc. 33rd AIAA Joint Propulsion Conference*, number 97-3051, Seattle, WA, 1997.
- ⁸S. Barral, K. Makowski, Z. Peradzyński, N. Gascon, and M. Dudeck. Wall material effects in stationary plasma thrusters. II. Near-wall and in-wall conductivity. *Phys. Plasmas*, 10:4137, 2003.
- ⁹J. C. Adam, A. Héron, and G. Laval. Study of stationary plasma thrusters using two-dimensional fully kinetic simulations. *Phys. Plasmas*, 11:295, 2004.
- ¹⁰J. Bareilles, G. J. M. Hagelaar, L. Garrigues, C. Boniface, and J. P. Boeuf. Critical assesment of a two-dimensional hybrid Hall thruster model: Comparisons with experiments. *Phys. Plasmas*, 11:3035, 2004.