

# Mathematical Model of Acceleration Stage of Magnetic Inductive Pulsed Plasma Thruster

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**Abstract:** Mathematical model of acceleration mechanism in magnetic inductive pulsed plasma thruster and possible modes of operation are presented. The mathematical model based on quasi-one dimensional model. Plasma is considered as a plasma sheet with own boundary conditions. One-component magneto hydrodynamic model and two-component plasma dynamic model are used as basic equations in Mathematical model.

## Nomenclature

|             |                                          |
|-------------|------------------------------------------|
| $\vec{A}$   | = electromagnetic field vector potential |
| $\vec{B}$   | = magnetic induction                     |
| $\vec{E}$   | = electric field tension                 |
| $\vec{j}$   | = current density                        |
| $J$         | = kinetic tensor charge analogue         |
| $L$         | = inductivity, mutual inductivity        |
| $m_e$       | = electron mass                          |
| $m_i$       | = ion mass                               |
| $n_e$       | = plasma components densities            |
| $p$         | = plasma bulk motion                     |
| $R$         | = resistance                             |
| $t$         | = time                                   |
| $T_e$       | = electron temperature                   |
| $T_i$       | = ion temperature                        |
| $v_e$       | = mean electrons velocity module         |
| $\vec{V}$   | = plasma mass flow velocity              |
| $\vec{V}_e$ | = electrons mass flow velocity           |
| $\vec{V}_i$ | = ions mass flow velocity                |
| $\vec{q}$   | = plasma energy flow density             |
| $\vec{q}_e$ | = electrons energy flow density          |
| $\vec{q}_i$ | = ions energy flow density               |
| $\delta$    | = unitary (trivial) tensor               |
| $\Pi$       | = plasma kinetic tensor                  |

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## I. Introduction

Pulse plasma thrusters is the sort of more common class of magnetic plasma dynamics accelerators where axial force, which accelerates plasma and generates the thrust, appears because of interaction of plasma currents and magnetic field with separate particles movement similar to Hall's drift.

In axial symmetric geometry axial acceleration of plasma can take place in two possible cases:

- Azimuth magnetic field  $B_\phi$  with radial electric field  $E_r$ ;
- Azimuth electric field  $E_\phi$  with radial magnetic field  $B_r$ .

The first combination is possible when electrodes are made as rotation figures – cones, cylinders. Azimuth magnetic field here is generated by axial current inside the central electrode (usually – cathode). Accelerators with this geometry can operate both in stationary and in pulse modes.

The second combination is possible with solenoids system use. Magnetic field here must necessary be variable in time because of the fact that azimuth (closed) electric field can be generated by only variable in time magnetic field.

This paper is purposed to description of processes inside magnetic inductive pulse plasma thruster, which relates to the last acceleration represented above.

Pulse inductive acceleration mode is represented on Fig. 1. Main source of pulse acceleration is the inductor – magnetic coil feeding by condensers battery. After condenser-inductor circuit switch-on variable current  $I_0$  is generated inside the inductor. This current induces variable axial symmetric magnetic field, which in turn induces variable azimuth electric field. The last one generates azimuth current  $I_p$  in plasma. Plasma appears inside the acceleration channel from ionizer where neutral gas is ionized by gas discharge under action fir example of high frequency electromagnetic oscillations.

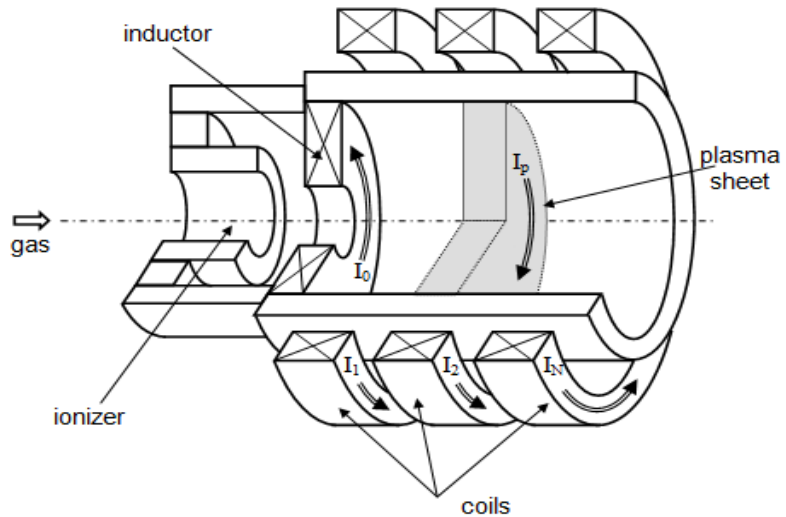


Figure 1. Magnetic inductive pulse plasma thruster structure.

The axial force, which accelerates plasma outside thruster channel, appears as a result of plasma azimuth current interaction with radial component of inductor magnetic field. Thus the acceleration mode in magnetic inductive pulse plasma thruster (MIPTT) is very similar to co-axial pulse plasma thruster operation mode. The only difference is between field and currents geometry and the principle difference lays in the absence of inter electrode discharge in MIPPT (as well as the absence of electrodes indeed). This fact eliminates the necessity of electrons emission in acceleration channel – surface temperature in MIPPT may be low.

One more structure element of MIPPT shown on Fig.1 is magnetic nozzle – the system of external coils feeding by direct voltage. Main purpose of this system is to protect acceleration channel wall from heating and ion bombardment. Also magnetic nozzle can generate additional accelerative force in plasma.

MIPPT structure permits to organize both the pulse and the quasi-stationary operation modes. The last one means the feeding of inductor by sinusoidal voltage when plasma is accelerated by separate portions moving inside thruster channel closely one after another. This mode permits to operate with constant mass flow rate as well as with

constant ionizer productivity.

But this work is purposed to pulse operation mode description when gas feed system and ionizer also operate in pulse regimes.

## II. M I P P T MATHEMATICAL MODEL

### Two-component plasma dynamics equations set (Ref. 1)

One-component magnetic hydrodynamics (MHD) equations set can be obtained on the base of two-components plasma dynamics, which describes plasma formed by electrons and single-charge positive ions. Two-components plasma dynamics equations set includes:

- substance equations:

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{V}_e) = \frac{\delta n_e}{\delta t}, \quad (1)$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{V}_i) = \frac{\delta n_i}{\delta t}; \quad (2)$$

where  $\frac{\delta n_{e,i}}{\delta t}$  – electrons and ions densities change per unit of time in collisions;

- motion equations:

$$m_e \frac{\partial}{\partial t} (n_e \vec{V}_e) + \nabla \cdot \mathbf{\Pi}_e + e n_e (\vec{E} + \vec{V}_e \times \vec{B}) = \frac{\delta \vec{p}_e^{(v)}}{\delta t}, \quad (3)$$

$$m_i \frac{\partial}{\partial t} (n_i \vec{V}_i) + \nabla \cdot \mathbf{\Pi}_i - e n_i (\vec{E} + \vec{V}_i \times \vec{B}) = \frac{\delta \vec{p}_i^{(v)}}{\delta t}; \quad (4)$$

where  $\frac{\delta \vec{p}_{e,i}^{(v)}}{\delta t}$  – motion density change per unit of time in collisions;

- energy equations:

$$\frac{\partial}{\partial t} \left[ n_e \left( \frac{m_e V_e^2}{2} + \frac{3}{2} k T_e \right) \right] + \nabla_e + e n_e \vec{V}_e \cdot \vec{E} = \frac{\delta \mathcal{E}_e^{(v)}}{\delta t}, \quad (3)$$

$$\frac{\partial}{\partial t} \left[ n_i \left( \frac{m_i V_i^2}{2} + \frac{3}{2} k T_i \right) \right] + \nabla \cdot \vec{q}_i - e n_i \vec{V}_i \cdot \vec{E} = \frac{\delta \mathcal{E}_i^{(v)}}{\delta t}. \quad (4)$$

where  $\frac{\delta \mathcal{E}_{e,i}^{(v)}}{\delta t}$  – energy density change per unit of time in collisions.

Motion and energy flow densities in non-dissipative approximation:

$$\mathbf{\Pi}_{e,i} = m_{e,i} n_{e,i} \vec{V}_{e,i} \vec{V}_{e,i} + \delta n_{e,i} k T_{e,i}, \quad (7)$$

$$\vec{q}_{e,i} = n_{e,i} \vec{V}_{e,i} \left( \frac{m_{e,i} V_{e,i}^2}{2} + \frac{5}{2} k T_{e,i} \right), \quad (8)$$

where  $\delta$  – unitary tensor:

$$\delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}. \quad (9)$$

### One-component magnetic hydrodynamics equations set

Frankly telling, the description of multi-component substance as one-component one permits to reduce the number of parameters (and consequently – the number of equations) only for entire equilibrium between the components – both by temperatures and mass flow velocities. According to plasma the existence of such equilibrium would by the way mean the absence of current – components flow densities difference.

The approximation of one-component MHD is the compromise, which permits to simplify the description only in several particular cases, including also the flows inside magnetic plasma dynamics thrusters – both stationary and pulse ones.

It is possible to pass from components mass flow velocities to plasma mass flow velocity  $\vec{V}$  and current density  $\vec{j}$ :

$$m_i \vec{V}_i + m_e \vec{V}_e = (m_i + m_e) \vec{V}, \quad (10)$$

$$e n_e (\vec{V}_i - \vec{V}_e) = \vec{j}. \quad (11)$$

Summary temperature is:

$$T_i + T_e = T. \quad (12)$$

It is possible to write from (10), (11) for  $\vec{V}_e$  and  $\vec{V}_i$ :

$$\vec{V}_i = \vec{V} + \frac{m_e}{m_i + m_e} \frac{\vec{j}}{e n_e}, \quad (13)$$

$$\vec{V}_e = \vec{V} - \frac{m_i}{m_i + m_e} \frac{\vec{j}}{e n_e}. \quad (14)$$

Summary and difference of substance equations (1), (2) give with consider of (10), (11):

- One-component MHD substance equation:

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{V}) = \frac{\delta n_e}{\delta t}; \quad (15)$$

- One-component MHD charge conservation law:

$$\nabla \cdot \vec{j} = 0. \quad (16)$$

Summary and difference (with division by mass in the last case) of motion equations (3), (4) give with

consider of (10), (11):

- One-component MHD motion equation:

$$(m_i + m_e) \frac{\partial}{\partial t} (n_e \vec{V}) + \nabla \cdot \mathbf{\Pi} - \vec{j} \times \vec{B} = \frac{\delta \vec{p}^{(v)}}{\delta t}; \quad (17)$$

- One-component MHD Ohm's law:

$$\frac{\partial \vec{j}}{\partial t} + \nabla \cdot \mathbf{J} - \frac{e^2 n_e (m_i + m_e)}{m_i m_e} (\vec{E} + \vec{V} \times \vec{B}) + \frac{e(m_i - m_e)}{m_i m_e} \vec{j} \times \vec{B} = \frac{\delta \vec{j}}{\delta t}. \quad (18)$$

Plasma motion and current densities change in collisions are equal:

$$\frac{\delta \vec{p}^{(v)}}{\delta t} = \frac{\delta \vec{p}_i^{(v)}}{\delta t} + \frac{\delta \vec{p}_e^{(v)}}{\delta t}, \quad (19)$$

$$\frac{\delta \vec{j}}{\delta t} = \frac{e}{m_i} \frac{\delta \vec{p}_i^{(v)}}{\delta t} - \frac{e}{m_e} \frac{\delta \vec{p}_e^{(v)}}{\delta t}. \quad (20)$$

Plasma kinetic tensor  $\mathbf{\Pi}$  and it's charge analogue  $\mathbf{J}$  are:

$$\mathbf{\Pi} = \mathbf{\Pi}_i + \mathbf{\Pi}_e, \quad (21)$$

$$\mathbf{J} = e \left( \frac{\mathbf{\Pi}_i}{m_i} - \frac{\mathbf{\Pi}_e}{m_e} \right). \quad (22)$$

One can see that substance and charge conservation equations (15), (16) are independent on each other. Only mass transport is represented in (15) and only charge transport – in (16). The limitation of one-component MHD method use is determined by the fact that unlike (15), (16) motion equation (17) and Ohm's law (18) are not independent ones. Both these equations include in very complex form the descriptions of both mass and charge transport.

For example for components kinetic tensors it follows from (7), (13), (14):

$$\mathbf{\Pi}_i = m_i n_e \vec{V} \vec{V} + \frac{m_e m_i}{m_i + m_e} \frac{\vec{V} \vec{j} + \vec{j} \vec{V}}{e} + \frac{m_e^2 m_i}{(m_i + m_e)^2} \frac{\vec{j} \vec{j}}{e^2 n_e} + \delta_{n_e k} T_i, \quad (23)$$

$$\mathbf{\Pi}_e = m_e n_e \vec{V} \vec{V} - \frac{m_e m_i}{m_i + m_e} \frac{\vec{V} \vec{j} + \vec{j} \vec{V}}{e} + \frac{m_e m_i^2}{(m_i + m_e)^2} \frac{\vec{j} \vec{j}}{e^2 n_e} + \delta_{n_e k} T_e. \quad (24)$$

So the expressions for  $\mathbf{\Pi}$  and  $\mathbf{J}$  are:

$$\mathbf{\Pi} = (m_i + m_e) n_e \vec{V} \vec{V} + \frac{m_e m_i}{m_i + m_e} \frac{\vec{j} \vec{j}}{e^2 n_e} + \delta_{n_e k} (T_e + T_i), \quad (25)$$

$$\mathbf{J} = \vec{V} \vec{j} + \vec{j} \vec{V} - \frac{m_i - m_e}{m_i + m_e} \frac{\vec{j} \vec{j}}{en_e} - \delta en_e k \left( \frac{T_e}{m_e} - \frac{T_i}{m_i} \right). \quad (26)$$

Sufficient simplification of one-component MHD expressions is possible with consider of both the universal property:

$$m_e \ll m_i, \quad (27)$$

and the specifics of flows in magnetic plasma dynamics thrusters:

- mass and charge transport in almost crossed directions:

$$\vec{j} \cdot \vec{V} \approx 0; \quad (28)$$

- mass transport velocity enough less then charge transport one:

$$V \ll \frac{j}{en_e}; \quad (29)$$

- however, mass transport velocity enough large to be possible to consider:

$$V \ll \sqrt{\frac{m_e}{m_i}} \frac{j}{en_e}. \quad (30)$$

Considering (30) it is possible to simplify the expressions (13), (14):

$$\vec{V}_i \approx \vec{V}, \quad (31)$$

$$\vec{V}_e \approx \vec{V} - \frac{\vec{j}}{en_e}. \quad (32)$$

Consideration of (28), (30) gives in (25), (26):

$$\mathbf{\Pi} \approx m_i n_e \vec{V} \vec{V} + \delta n_e k T, \quad (33)$$

$$\mathbf{J} \approx \vec{V} \vec{j} + \vec{j} \vec{V} - \frac{\vec{j} \vec{j}}{en_e} - \delta \frac{en_e k T_e}{m_e}, \quad (34)$$

while the motion equation (17) and Ohm's law (18) obtain the forms:

$$m_i \frac{\partial}{\partial t} (n_e \vec{V}) + \nabla \cdot \mathbf{\Pi} - \vec{j} \times \vec{B} \approx \frac{\delta \vec{p}^{(v)}}{\delta t}, \quad (35)$$

$$\frac{\partial \vec{j}}{\partial t} + \nabla \cdot \mathbf{J} - \frac{e^2 n_e}{m_e} (\vec{E} + \vec{V} \times \vec{B}) + \frac{e}{m_e} \vec{j} \times \vec{B} \approx \frac{\delta \vec{j}}{\delta t}. \quad (36)$$

It is possible to see that under shown conditions tensor  $\mathbf{\Pi}$  (33) includes only mass transport descriptions.

Vice versa, tensor  $\mathbf{J}$  (34) includes the descriptions of both transports together with electron pressure. The last fact means the necessity to remain separately electrons energy equation for electrons temperature  $T_e$  search together with plasma energy equation (the summary of partial ones) for summary temperature  $T$ .

There, considering the non-equivalences shown before, it is possible to state:

$$m_e V_e^2 \approx \frac{m_e j^2}{e^2 n_e^2}, \quad (37)$$

$$m_i V_i^2 + m_e V_e^2 \approx m_i V^2, \quad (38)$$

$$m_e \vec{V}_e V_e^2 \approx \left( \vec{V} - \frac{\vec{j}}{e n_e} \right) \frac{m_e j^2}{e^2 n_e^2}, \quad (39)$$

$$m_i \vec{V}_i V_i^2 + m_e \vec{V}_e V_e^2 \approx m_i \vec{V} V^2. \quad (40)$$

Thus from (5), (6) it is possible to write entire plasma and electron component energy equations:

$$\frac{\partial}{\partial t} \left[ n_e \left( \frac{m_i V^2}{2} + \frac{3}{2} k T \right) \right] + \nabla \cdot \vec{q} - \vec{j} \cdot \vec{E} = \frac{\delta \mathcal{E}^{(v)}}{\delta t}, \quad (41)$$

$$\frac{\partial}{\partial t} \left[ n_e \left( \frac{m_e j^2}{2 e^2 n_e^2} + \frac{3}{2} k T_e \right) \right] + \nabla \cdot \vec{q}_e - \vec{j} \cdot \vec{E} = \frac{\delta \mathcal{E}_e^{(v)}}{\delta t}, \quad (42)$$

where plasma energy density change in collisions is equal to:

$$\frac{\delta \mathcal{E}^{(v)}}{\delta t} = \frac{\delta \mathcal{E}_i^{(v)}}{\delta t} + \frac{\delta \mathcal{E}_e^{(v)}}{\delta t}, \quad (43)$$

electrons energy flow density:

$$\vec{q}_e \approx n_e \vec{V} \left( \frac{m_e j^2}{2 e^2 n_e^2} + \frac{5}{2} k T_e \right) - \frac{\vec{j}}{e} \left( \frac{m_e j^2}{2 e^2 n_e^2} + \frac{5}{2} k T_e \right), \quad (44)$$

and plasma energy flow density:

$$\vec{q} \approx n_e \vec{V} \left( \frac{m_i V^2}{2} + \frac{5}{2} k T \right) - \frac{5}{2} \frac{\vec{j}}{e} k T_e. \quad (45)$$

Considering (16) and neglecting all parameters change in current direction (crossed with mass transport direction) it is possible to write

$$\nabla \cdot \vec{q} \approx \nabla \cdot \left[ n_e \vec{V} \left( \frac{m_i V^2}{2} + \frac{5}{2} k T \right) \right], \quad (46)$$

$$\nabla \cdot \vec{q}_e \approx \nabla \cdot \left[ n_e \vec{V} \left( \frac{m_e j^2}{2 e^2 n_e^2} + \frac{5}{2} k T_e \right) \right]. \quad (47)$$

Small mass of electrons permits to reduce complex in (42), (47). So it is possible to write:

$$\frac{\partial}{\partial t} \left[ n_e \left( \frac{m_i V^2}{2} + \frac{3}{2} k T \right) \right] + \nabla \cdot \left[ n_e \vec{V} \left( \frac{m_i V^2}{2} + \frac{5}{2} k T \right) \right] - \vec{j} \cdot \vec{E} = \frac{\delta \varepsilon^{(v)}}{\delta t}, \quad (48)$$

$$\frac{\partial}{\partial t} \left( \frac{3}{2} n_e k T_e \right) + \nabla \cdot \left( n_e \vec{V} \frac{5}{2} k T_e \right) - \vec{j} \cdot \vec{E} = \frac{\delta \varepsilon_e^{(v)}}{\delta t}, \quad (49)$$

One-component MHD equations set including:

- substance equation (15);
- charge conservation law (16);
- motion equation (35);
- Ohm's law (36);
- plasma energy equation (48);
- electrons energy equation (49)

must be combined with magnetic field rotor equation. Electric field change in time sufficiently reflects in magnetic field behavior only under relativistic conditions. So even in non-stationary but non-relativistic case it is possible to use stationary form of rotor equation:

$$\varepsilon_0 c^2 \nabla \times \vec{B} = \vec{j}. \quad (50)$$

### Initial equations set

Mathematics model of plasma acceleration in MIPPT includes two equations sets:

- the equations, which describe electromagnetic field behavior:

- electric field tension  $\vec{E}$  and magnetic induction  $\vec{B}$  relations with scalar  $\varphi$  and vector potentials  $\vec{A}$  :

$$\vec{E} = -\nabla \varphi - \frac{\partial \vec{A}}{\partial t}, \quad (51)$$

$$\vec{B} = \nabla \times \vec{A}, \quad (52)$$

- Poisson's equation for vector potential:

$$\Delta \vec{A} = -\frac{\vec{j}}{\varepsilon_0 c^2}; \quad (53)$$

- one-component MHD (15), (16), (35), (36), (48), (49):

- substance equation :

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{V}) = 0, \quad (54)$$

- charge conservation law (in quasi-neutral volume):

$$\nabla \cdot \vec{j} = 0, \quad (55)$$

- plasma motion equation:

$$m_i \frac{\partial}{\partial t} (n_e \vec{V}) + m_i n_e \vec{V} \cdot \nabla \vec{V} + m_i \vec{V} \nabla \cdot (n_e \vec{V}) + \nabla (n_e k T) - \vec{j} \times \vec{B} = 0, \quad (56)$$

- one-component MHD Ohm's law:

$$\begin{aligned} \frac{\partial \vec{j}}{\partial t} + \nabla \cdot \left( \vec{V} \vec{j} + \vec{j} \vec{V} - \frac{\vec{j} \vec{j}}{e n_e} \right) - \frac{e}{m_e} \nabla (n_e k T_e) + \frac{e}{m_e} \vec{j} \times \vec{B} - \\ - \frac{e^2 n_e}{m_e} (\vec{E} + \vec{V} \times \vec{B}) = -n_e v_e \sigma_{ei} \vec{j} \end{aligned} \quad (57)$$

where  $v_e$  – mean electrons velocity module;

$\sigma_{ei}$  – mean electron-ion transport cross-section.

Plasma acceleration equations set (54) – (57) is obtained from (15), (16), (35), (36) for completely ionized plasma with consideration of motion exchange in electron-ions collisions.

### Quasi-one-dimension model

Let us insert the some parameter P mean-by-radius value:

$$P(x) = \frac{2}{r_p^2} \int_0^{r_p} P(x, r) r dr, \quad (58)$$

where  $x, r$  – axial and radial coordinates in cylindrical coordinate system;

$r_p$  – acceleration channel external boundary.

Any vector  $\vec{A}$  divergence under axial symmetry is equal to:

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (A_r r). \quad (59)$$

Integration (58) of (59) right part second component gives:

$$\frac{2}{r_p^2} \int_0^{r_p} \frac{1}{r} \frac{\partial}{\partial r} (A_r(x, r) r) r dr = \frac{2}{r_p} A_r(x, r_p). \quad (60)$$

Taking into consideration plasma flow on insulator radial boundary with ion-sonic velocity:

$$V_s = \sqrt{\frac{k T}{m_i}}, \quad (61)$$

we can write from (54), (56), (57) using (58) – (61):

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e V_x) + 2 \frac{\eta_r}{r_+} \sqrt{\frac{kT}{m_i}} n_e = 0, \quad (62)$$

$$m_i \frac{\partial}{\partial t}(n_e V_x) + \frac{\partial}{\partial x}[n_e(m_i V_x^2 + kT)] + 2 \frac{\eta_r}{r_+} \sqrt{\frac{kT}{m_i}} m_i n_e V_x = j_\psi B_r, \quad (63)$$

$$j_\psi = \sigma(E_\psi + V_x B_r). \quad (64)$$

where  $\eta_r$  – the relation of electrons concentration on external boundary to mean-by-radius one;  
 $\psi$  – azimuth coordinate in cylindrical coordinate system;  
 $\sigma$  – plasma electric conductivity:

$$\sigma = \frac{e^2}{m_e v_e \sigma_{ei}}. \quad (65)$$

Equation (63) can be written in such form:

$$m_i \frac{\partial}{\partial t}(n_e V_x) + m_i \frac{\partial}{\partial x}(n_e V_x^2) + \frac{\partial}{\partial x}(n_e kT) + 2 \frac{\eta_r}{r_+} \sqrt{\frac{kT}{m_i}} m_i n_e V_x = j_\psi B_r. \quad (66)$$

### Plasma sheet model

Plasma sheet model considers plasma like finite volume, which occupies any time small part of thruster channel length. Magnetic field and plasma density configuration inside MIPPT are shown on the fig. 2.

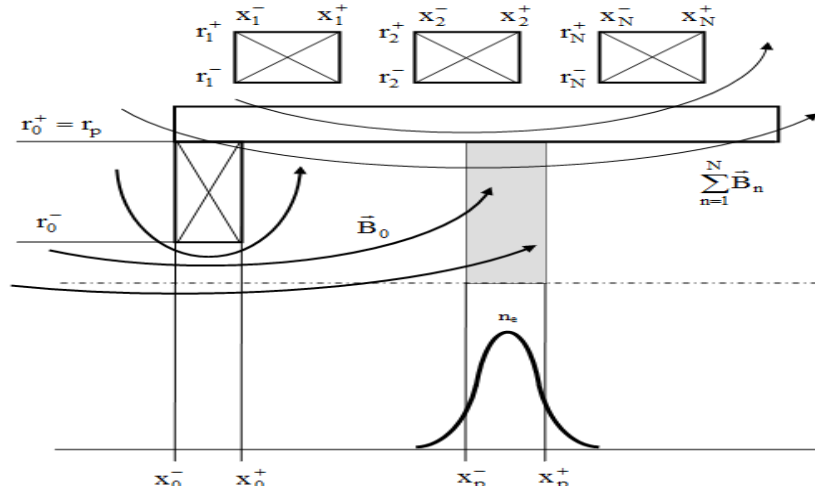


Figure 1. Magnetic field and plasma density configuration inside MIPPT.

Let us suppose that at the time  $t$  plasma occupies the axial coordinate range:

$$x_p^-(t) \leq x \leq x_p^+(t). \quad (67)$$

Let us also consider the pressure on sheet boundaries like equal to zero:

$$P_- = P_+ = 0. \quad (68)$$

Plasma sheet mass  $M$  and motion  $p$  are equal:

$$M = \pi r_p^2 m_i \int_{x_p^-}^{x_p^+} n_e dx, \quad (69)$$

$$p = MV = \pi r_p^2 m_i \int_{x_p^-}^{x_p^+} n_e V_x dx. \quad (70)$$

Any parameter  $A$  value inside plasma volume also can be found like integral of this parameter density  $A^{(v)}$ :

$$A = \pi r_p^2 \int_{x_p^-}^{x_p^+} A^{(v)} dx. \quad (71)$$

Parameter  $A$  full derivative by time can be written as:

$$\frac{dA}{dt} = \pi r_p^2 \left( \int_{x_p^-}^{x_p^+} \frac{\partial A^{(v)}}{\partial t} dx + \frac{dx_p^+}{dt} A_{x_p^+}^{(v)} - \frac{dx_p^-}{dt} A_{x_p^-}^{(v)} \right). \quad (72)$$

Considering the fact that plasma sheet axial boundaries changes per unitary time are equal to plasma mass flow velocities:

$$\frac{dx_p^\pm}{dt} = V_{x_p^\pm}, \quad (73)$$

it is possible to write from (72):

$$\frac{dA}{dt} = \pi r_p^2 \int_{x_p^-}^{x_p^+} \left[ \frac{\partial A^{(v)}}{\partial t} + \frac{\partial}{\partial x} (A^{(v)} V_x) \right] dx. \quad (74)$$

Considering (52), (68), (74) the expressions (62) and (66) can be transformed into the next ones:

$$\frac{dM}{dt} + 2 \frac{\eta_r}{r_p} \sqrt{\frac{kT}{m_i}} M = 0, \quad (75)$$

$$\frac{d}{dt} (MV) + 2 \frac{\eta_r}{r_p} \sqrt{\frac{kT}{m_i}} MV = - \int_{V_p} j_\psi \frac{\partial A_\psi}{\partial x} dV. \quad (76)$$

Right part of (76) can be written in terms of electric engineering (see Appendix A):

$$- \int_{V_p} j_\psi \frac{\partial A_\psi}{\partial x} dV = I_p \sum_{n=0}^N \frac{dL_{pn}}{dx_p} I_n, \quad (77)$$

where  $I_p$  – the current inside the plasma sheet;  
 $I_n$  – the current inside n-magnet (inductor with  $n=0$  or external coil with  $1 \leq n \leq N$ );  
 $L_{pn}$  – Plasma sheet and n-magnet mutual inductivity;  
 $x_p = x_p^-$  – Plasma sheet actual coordinate (fig. 3.1).

The combination of (75) – (77) gives plasma sheet acceleration equation:

$$M \frac{dV}{dt} = I_p \sum_{n=0}^N \frac{dL_{pn}}{dx_p} I_n, \quad (78)$$

which must be used together with Ohm's law for plasma sheet:

$$L_{pp} \frac{dI_p}{dt} + \sum_{n=0}^N \frac{d}{dt} (L_{pn} I_n) + I_p R_p = 0 \quad (79)$$

For each external coil and for inductor (see Appendix A):

$$\frac{d}{dt} (L_{pn} I_p) + \sum_{m=0}^N L_{mn} \frac{dI_m}{dt} + I_n R_n = U_n \quad (80)$$

and charge conservation law:

$$I_0 = -C \frac{dU_0}{dt}, \quad (81)$$

Where  $U_n$  – the voltage on inductor condensers ( $n=0$ ) and external coil voltage source ( $1 \leq n \leq N$ );  
 $C$  – Inductor condensers capacity;  
 $L_{pp}$  – Plasma sheet inductivity;  
 $L_{nm}$  – n-magnet and m-magnet ( $n \neq m$ ) mutual inductivity;  
 $L_{nn}$  – The inductivity of n-magnet;  
 $R_p$  – Plasma sheet resistance;  
 $R_n$  – n-magnet electric circuit resistance.

Considering the relation

$$\frac{dx_p}{dt} = V, \quad (82)$$

Plasma sheet and n-magnet mutual inductivity derivative by time can be found from the following expression:

$$\frac{dL_{pn}}{dt} = V \frac{dL_{pn}}{dx_p}. \quad (83)$$

### Energy transfer during acceleration process

The product of (79) and plasma sheet current gives the expression:

$$L_{pp}I_p \frac{dI_p}{dt} + I_p \sum_{n=0}^N L_{pn} \frac{dI_n}{dt} + I_p \sum_{n=0}^N I_n \frac{dL_{pn}}{dt} + I_p^2 R_p = 0. \quad (84)$$

Also the product of (81) by n-magnet with following summary for all n-magnet gives the expression:

$$\sum_{n=0}^N I_n \frac{d}{dt} (L_{pn} I_p) + \sum_{n=0}^N \sum_{m=0}^N L_{mn} I_n \frac{dI_m}{dt} + \sum_{n=0}^N I_n^2 R_n = \sum_{n=0}^N I_n U_n. \quad (85)$$

The following symmetry will be shown later in Appendix A:

$$L_{mn} = L_{nm}. \quad (86)$$

Considering (86) it is possible to write:

$$\begin{aligned} \sum_{n=0}^N \sum_{m=0}^N L_{mn} I_n \frac{dI_m}{dt} &= \sum_{n=0}^N \sum_{m=0}^N L_{mn} I_m \frac{dI_n}{dt} = \\ &= \frac{1}{2} \sum_{n=0}^N \sum_{m=0}^N L_{mn} \left( I_n \frac{dI_m}{dt} + I_m \frac{dI_n}{dt} \right) = \frac{d}{dt} \sum_{n=0}^N \sum_{m=0}^N \frac{L_{mn} I_n I_m}{2}. \end{aligned} \quad (87)$$

The combination of (78) and (83) gives the expression:

$$MV \frac{dV}{dt} = I_p \sum_{n=0}^N \frac{dL_{pn}}{dt} I_n, \quad (88)$$

The summary of (84), (85) with consider of (75), (81), (87) and (88) gives the energy transfer equation:

$$\frac{d}{dt} (\mathcal{E}_e + \mathcal{E}_m + \mathcal{E}_k) = W_c - W_R - W_k \quad (89)$$

Where  $\mathcal{E}_e$ ,  $\mathcal{E}_m$ ,  $\mathcal{E}_k$  – electric and magnetic energies and plasma sheet kinetic energy;

$W_c$ ,  $W_R$ ,  $W_k$  – external coils power consumption, resistive energy lost per second and plasma sheet kinetic energy lost per second:

$$\mathcal{E}_e = \frac{CU_0^2}{2}, \quad (90)$$

$$\mathcal{E}_m = \frac{L_{pp}I_p^2}{2} + \sum_{n=0}^N L_{pn} I_p I_n + \sum_{n=0}^N \sum_{m=0}^N \frac{L_{mn} I_n I_m}{2}, \quad (91)$$

$$\mathcal{E}_k = \frac{MV^2}{2}; \quad (92)$$

$$W_c = \sum_{n=1}^N I_n U_n, \quad (93)$$

$$W_R = I_p^2 R_p + \sum_{n=0}^N I_n^2 R_n, \quad (94)$$

$$W_k = \frac{\eta_r}{r_p} \sqrt{\frac{kT}{m_i}} M V^2. \quad (95)$$

The expressions (90), (93) will be considered in Appendix A while acceleration process electric engineering coefficients definition.

### Energy equations

The equations set (78) – (81) is very similar to so called Artsimovitch's equations set for pulse plasma thrusters with axial symmetric magnetic field. Also like Artsimovitch's one the equations set (78) – (81) must be completed by substance equation (75) together with plasma and electrons energy equations. Energy equations can be obtained from (48), (49) considering energy lost in ionization processes.

Quasi-one-dimension form of energy equation in electric engineering terms is as follows:

$$M \frac{d}{dt} \left( \frac{V^2}{2} + \frac{3}{2} \frac{kT}{m_i} + \frac{e\varphi_i}{m_i} \right) = I_p V \sum_{n=0}^N \frac{dL_{pn}}{dx_p} I_n, \quad (96)$$

$$M \frac{d}{dt} \left( \frac{V^2}{2} + \frac{3}{2} \frac{kT}{m_i} + \frac{e\varphi_i}{m_i} \right) = I_p V \sum_{n=0}^N \frac{dL_{pn}}{dx_p} I_n. \quad (97)$$

### Zero-approximation

The most common features of MIPPT operation can be demonstrated by simplified (78) – (83) equations set without consideration of external magnets:

$$M \frac{dV}{dt} = \frac{dL_{p0}}{dx_p} I_p I_0, \quad (98)$$

$$L_{pp} \frac{dI_p}{dt} + \frac{d}{dt} (L_{p0} I_0) + I_p R_p = 0 \quad (99)$$

$$\frac{d}{dt} (L_{p0} I_p) + L_{00} \frac{dI_0}{dt} + I_0 R_0 = U_n \quad (100)$$

$$I_0 = -C \frac{dU_0}{dt}, \quad (101)$$

$$\frac{dx_p}{dt} = V, \quad (102)$$

$$\frac{dL_{pn}}{dt} = V \frac{dL_{pn}}{dx_p}. \quad (103)$$

The analysis of (98) – (103) equations set shows that condenser voltage, inductor and plasma sheet currents

behavior in time is of oscillation nature with decreasing magnitudes (see Fig. 3). Depending on condenser capacity, inductor and plasma sheet self and mutual inductivities the acceleration process during one pulse can include several or less than one oscillations period. For example, the situation shown on fig. 3 relates to the case of 2.5 semi-periods of oscillations when the voltage and both the currents change the sign two times.

The length of acceleration channel must be chosen considering the time of effective acceleration when considerable acceleration yet takes place.

#### External coils action

The action of external coils can result in two different effects:

- to protect acceleration channel external wall from direct action of hot plasma as well as from fast ions bombardment;
- to produce additive plasma sheet acceleration.

The first effect is determined by axial component of external coils magnetic field. The second effect is determined by radial component of magnetic field. Depending on combination of  $I_p$  and  $B_r$  signs the sign of additional acceleration can be different. For example, the combination shown on Fig. 4 demonstrates the possibility to provide the situation when  $I_p$  and external  $B_r$  change the sign in the same time – additional acceleration remains to be positive via entire acceleration channel length.

So the variation of external coils geometry and feed voltage can provide the combination with the most effective additional acceleration inside magnetic inductive pulse plasma thruster.

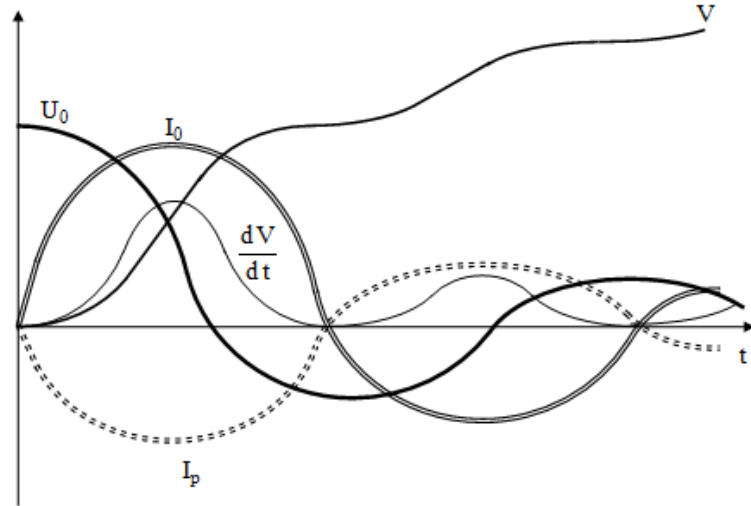


Figure 3. MIPPT parameters change in time during one pulse.

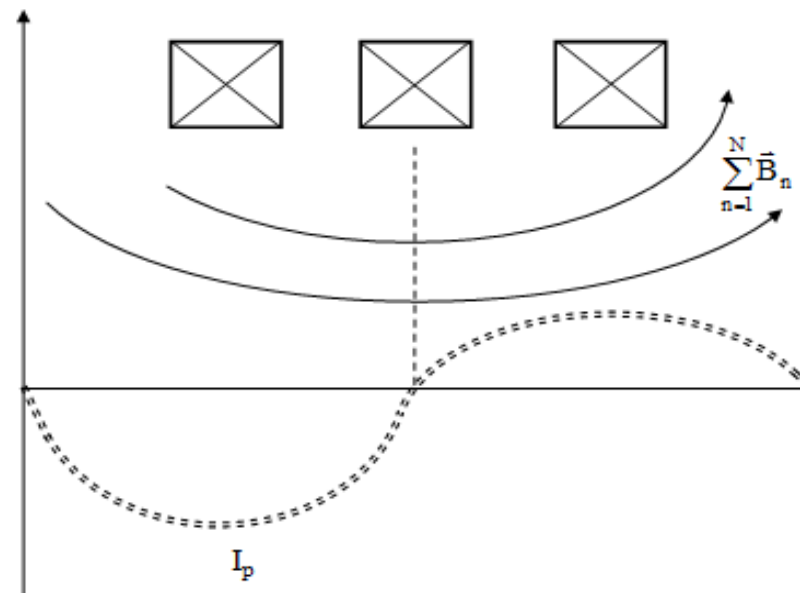


Figure 4. Correlation in  $I_p$  and  $B_r$  sign.

### III Conclusion

The operation mode described higher is not the only possible for shown accelerator structure. Also quasi-stationary operation mode is considered to be realized with harmonic voltage for inductor power supply. Gas feeding and ionization system here can operate in stationary regime.

The mathematical model of quasi-stationary operation mode is now under development with Fourier series use describing all the parameters like the series of inductor feed voltage frequency harmonics.

### APPENDIX

#### MIPPT electric engineering coefficients (Ref. 2)

Insertion of (5) into (64) gives the expression (index  $\psi$  in further expression will not be written, considering only azimuth direction of current):

$$j = \sigma \left( -\frac{1}{r} \frac{\partial \varphi}{\partial \psi} - \frac{\partial A_\psi}{\partial t} - V_x \frac{\partial A_\psi}{\partial x} \right). \quad (\text{A.1})$$

Multiplying (A.1) by  $j$  and using the definition of full derivative by time it is possible to write:

$$-j \frac{1}{r} \frac{\partial \varphi}{\partial \psi} = j \frac{d A_\psi}{d t} + \frac{1}{\sigma} j^2. \quad (\text{A.2})$$

Integration of (A.2) by the volume occupied by current  $j$  gives the integral form of Ohm's law:

$$-\int_V j \frac{1}{r} \frac{\partial \varphi}{\partial \psi} dV = \int_V j \frac{d A_\psi}{d t} dV + \frac{1}{\sigma} \int_V j^2 dV, \quad (\text{A.3})$$

where

$$dV = r dr dx d\psi. \quad (\text{A.4})$$

So finally we have:

$$\begin{aligned} & - \int_{x^-}^{x^+} \int_{r^-}^{r^+} \int_0^{2\pi} j_\psi \frac{\partial \varphi}{\partial \psi} r dr dx d\psi = \\ & = \int_{x^-}^{x^+} \int_{r^-}^{r^+} \int_0^{2\pi} j_\psi \frac{d A_\psi}{d t} r dr dx d\psi + \int_{x^-}^{x^+} \int_{r^-}^{r^+} \int_0^{2\pi} j_\psi^2 r dr dx d\psi \end{aligned} \quad (\text{A.5})$$

If  $V$  is the volume of electromagnet coil it is possible to show the voltage drop via one loop:

$$- \int_0^{2\pi} \frac{\partial \varphi}{\partial \psi} d\psi = U \quad (\text{A.6})$$

as well as for coil current in Ampere-loops:

$$\int_{x^-}^{x^+} \int_{r^-}^{r^+} j dr dx = I. \quad (\text{A.7})$$

Expression (A.5) shows the division of coil feed power  $IU$  into magnetic power  $W^{(M)}$  and resistive losses  $W_R$ :

$$IU = W^{(M)} + W_R, \quad (\text{A.8})$$

where:

$$W_R = \frac{1}{\sigma} \int_{x^-}^{x^+} \int_{r^-}^{r^+} \int_0^{2\pi} j^2 r dr dx d\psi, \quad (\text{A.9})$$

$$W^{(M)} = \int_{x^-}^{x^+} \int_{r^-}^{r^+} \int_0^{2\pi} j \frac{dA_\psi}{dt} r dr dx d\psi. \quad (\text{A.10})$$

Electric engineering expression for  $W_R$  is well known:

$$W_R = I^2 R. \quad (\text{A.11})$$

So we can write for n-magnet and plasma sheet resistance (fig. 1):

$$R_n = \frac{2\pi \int_{x_n^-}^{x_n^+} \int_{r_n^-}^{r_n^+} j_n^2 r_n dr_n dx_n}{\sigma \left( \int_{x_n^-}^{x_n^+} \int_{r_n^-}^{r_n^+} j_n dr_n dx_n \right)^2}, \quad R_p = \frac{2\pi \int_{x_p^-}^{x_p^+} \int_{r_p^-}^{r_p^+} j_p^2 r_p dr_p dx_p}{\sigma \left( \int_{x_p^-}^{x_p^+} \int_{r_p^-}^{r_p^+} j_p dr_p dx_p \right)^2}. \quad (\text{A.14})$$

General solution of Poisson's equation (53) for vector potential  $\vec{A}$  is:

$$\vec{A}(\vec{r}) = \frac{1}{4\pi \varepsilon_0 c^2} \int_{V'} \frac{\vec{j}(\vec{r}') dV'}{|\vec{r} - \vec{r}'|} = \frac{1}{4\pi \varepsilon_0 c^2} \int_{V'} \frac{i_\psi(\vec{r}') j(\vec{r}') dV'}{|\vec{r} - \vec{r}'|}. \quad (\text{A.15})$$

Azimuth component of  $\vec{A}$  in any point  $\vec{r}$  is:

$$\begin{aligned} A_\psi(\vec{r}) &= \frac{1}{4\pi \varepsilon_0 c^2} \int_{V'} \frac{i_\psi(\vec{r}) \cdot i_\psi(\vec{r}') j(\vec{r}') dV'}{|\vec{r} - \vec{r}'|} = \\ &= \frac{1}{4\pi \varepsilon_0 c^2} \int_{V'} \frac{j(\vec{r}') \cos \Delta\psi dV'}{\sqrt{(x' - x)^2 + r'^2 + r^2 - 2r'r \cos \Delta\psi}}, \end{aligned} \quad (\text{A.16})$$

where  $i_\psi(\vec{r}')$ ,  $i_\psi(\vec{r})$  – azimuth orts in points  $\vec{r}'$  and  $\vec{r}$ ;

$\Delta\Psi$  – the angle between the orts  $i_\psi(\vec{r}')$ ,  $i_\psi(\vec{r})$ .

If current occupies some separate volumes  $V_m$  it is possible to write:

$$A_\psi(\vec{r}) = \sum_m A_{\psi,m}(\vec{r}), \quad (\text{A.17})$$

where

$$A_{\psi,m}(\vec{r}) = \frac{1}{4\pi\epsilon_0 c^2} \int_{x_m^-}^{x_m^+} \int_{r_m^-}^{r_m^+} \int_0^{2\pi} \frac{j(\vec{r}_m) \cos \Delta\Psi r_m dr_m dx_m d\Delta\Psi}{\sqrt{(x_m - x)^2 + r_m^2 + r^2 - 2r_m r \cos \Delta\Psi}}. \quad (\text{A.18})$$

Thus (A.8) for separate volume  $V_m$  can be written as follows:

$$W_n^{(M)} = \sum_m W_{nm}^{(M)}, \quad (\text{A.19})$$

where

$$W_{nm}^{(M)} = \int_{x_n^-}^{x_n^+} \int_{r_n^-}^{r_n^+} \int_0^{2\pi} j(\vec{r}_n) \frac{dA_{\psi,m}(\vec{r}_n)}{dt} r_n dr_n dx_n d\psi_n. \quad (\text{A.20})$$

The comparison of (A.18) and (A.20) with (3.30) permits to write:

$$W_{nm}^{(M)} = I_n \frac{d}{dt} (L_{mn} I_m), \quad (\text{A.21})$$

where mutual inductivity of n-volume and m-volume is:

$$L_{nm} = \frac{\int_{x_n^-}^{x_n^+} \int_{r_n^-}^{r_n^+} \int_0^{2\pi} \int_{x_m^-}^{x_m^+} \int_{r_m^-}^{r_m^+} \int_0^{2\pi} \frac{j_n j_m \cos \Delta\Psi r_n dr_n dx_n dr_m dx_m d\Delta\Psi}{\sqrt{(x_m - x_n)^2 + r_m^2 + r_n^2 - 2r_m r_n \cos \Delta\Psi}}}{2\epsilon_0 c^2 \int_{x_m^-}^{x_m^+} \int_{r_m^-}^{r_m^+} j_m dr_m dx_m \int_{x_n^-}^{x_n^+} \int_{r_n^-}^{r_n^+} j_n dr_n dx_n}. \quad (\text{A.22})$$

Partial integration by  $\Delta\Psi$  in (A.22) gives the expression:

$$L_{nm} = \frac{\pi}{8\epsilon_0 c^2} \times \frac{\int_{x_n^-}^{x_n^+} \int_{r_n^-}^{r_n^+} \int_{x_m^-}^{x_m^+} \int_{r_m^-}^{r_m^+} \Re \left( \frac{4r_m r_n}{(x_m - x_n)^2 + (r_m + r_n)^2} \right) j_n j_m r_m^2 r_n^2 dr_n dx_n dr_m dx_m}{\int_{x_m^-}^{x_m^+} \int_{r_m^-}^{r_m^+} j_m dr_m dx_m \int_{x_n^-}^{x_n^+} \int_{r_n^-}^{r_n^+} j_n dr_n dx_n} \frac{1}{[(x_m - x_n)^2 + (r_m + r_n)^2]^{\frac{3}{2}}}, \quad (\text{A.23})$$

where

$$\Re(p) = \frac{4}{\pi p} \int_0^{2\pi} \frac{\cos \Delta\psi}{\sqrt{1-p \frac{1+\cos \Delta\psi}{2}}} d\Delta\psi = \frac{16}{\pi p} \int_0^{\frac{\pi}{2}} \frac{(2\sin^2 \theta - 1) d\theta}{\sqrt{1-p \sin^2 \theta}}. \quad (\text{A.24})$$

The method of  $\Re(p)$  calculation is shown in Appendix B.

Mutual inductivity of n-magnet and plasma sheet can be written from (A.23):

$$L_{pn} = \frac{\pi}{8\epsilon_0 c^2} \times \frac{\int_{x_p^-}^{x_p^+} \int_{r_p^-}^{r_p^+} \int_{x_n^-}^{x_n^+} \int_{r_n^-}^{r_n^+} \Re \left( \frac{4r_p r_n}{(x_p - x_n)^2 + (r_p + r_n)^2} \right) j_n j_p r_p^2 r_n^2 d r_p d x_p d r_n d x_n}{\int_{x_p^-}^{x_p^+} \int_{r_p^-}^{r_p^+} j_p d r_p d x_p \int_{x_n^-}^{x_n^+} \int_{r_n^-}^{r_n^+} j_n d r_n d x_n} \quad (\text{A.25})$$

that also means

$$L_{pn} = -[\ell_{pn}(x_p^-) - \ell_{pn}(x_p^+)]. \quad (\text{A.26})$$

where

$$\ell_{pn}(x_p^\pm) = \frac{\pi}{8\epsilon_0 c^2} \times \frac{\int_{x_n^-}^{x_n^+} \int_{r_n^-}^{r_n^+} \Re \left( \frac{4r_p r_n}{(x_p^\pm - x_n)^2 + (r_p + r_n)^2} \right) j_p j_n r_p^2 r_n^2 d r_p d r_n d x_n}{\int_{x_p^-}^{x_p^+} j_p d r_p d x_p \int_{x_n^-}^{x_n^+} j_n d r_n d x_n}. \quad (\text{A.27})$$

Expressions (A.14), (A.23), (A.25) and (A.27) must be used with some suppositions about current density radial distribution inside the magnets and plasma sheet. The simplest but adequate to the content of processes are the following suppositions:

$$j_n = \text{Const}, \quad j_p \sim r. \quad (\text{A.28})$$

It means the following equivalences:

$$R_n = \frac{\pi(r_n^+ + r_n^-)}{\sigma(r_n^+ - r_n^-)(x_n^+ - x_n^-)}, \quad R_p = \frac{8\pi}{3\sigma(x_p^+ - x_p^-)}, \quad (\text{A.29})$$

$$L_{nm} = \frac{\pi \int_{x_n^-}^{x_n^+} \int_{r_n^-}^{r_n^+} \int_{x_m^-}^{x_m^+} \int_{r_m^-}^{r_m^+} \Re \left( \frac{4r_m r_n}{(x_m - x_n)^2 + (r_m + r_n)^2} \right) r_m^2 r_n^2 d r_n d x_n d r_m d x_m}{8\epsilon_0 c^2 (x_n^+ - x_n^-)(r_n^+ - r_n^-)(x_m^+ - x_m^-)(r_m^+ - r_m^-)}, \quad (\text{A.30})$$

$$L_{pn} = \frac{\pi \int_{x_p^-}^{x_p^+} \int_{x_n^-}^{x_n^+} \int_{r_n^-}^{r_n^+} \frac{\Re \left( \frac{4 r_p r_n}{(x_p - x_n)^2 + (r_p + r_n)^2} \right) r_p^3 r_n^2 d r_p d x_p d r_n d x_n}{\left[ (x_p - x_n)^2 + (r_p + r_n)^2 \right]^{\frac{3}{2}}}}{4 \varepsilon_0 C^2 (x_n^+ - x_n^-) (r_n^+ - r_n^-) (x_p^+ - x_p^-) r_p^2}, \quad (\text{A.31})$$

$$\ell_{pn}(x_p^\pm) = \frac{\pi \int_{x_p^-}^{x_p^+} \int_{x_n^-}^{x_n^+} \int_{r_n^-}^{r_n^+} \frac{\Re \left( \frac{4 r_p r_n}{(x_p^\pm - x_n)^2 + (r_p + r_n)^2} \right) r_p^3 r_n^2 d r_p d x_p d r_n d x_n}{\left[ (x_p - x_n)^2 + (r_p + r_n)^2 \right]^{\frac{3}{2}}}}{4 \varepsilon_0 C^2 (x_n^+ - x_n^-) (r_n^+ - r_n^-) (x_p^+ - x_p^-) r_p^2}. \quad (\text{A.32})$$

### Function $\Re(p)$ calculation

Function  $\Re(p)$  used for inductivities calculation is defined by expression (A.24):

$$\Re(p) = \frac{16}{\pi p} \int_0^{\frac{\pi}{2}} \frac{(2 \sin^2 \theta - 1) d \theta}{\sqrt{1 - p \sin^2 \theta}}. \quad (\text{A.33})$$

It is possible to show from (A.33) that  $\Re(p)$  is the solution of differential equation:

$$p(1-p) \frac{d^2 \Re(p)}{d p^2} + (3-4p) \frac{d \Re(p)}{d p} - \frac{9}{4} \Re(p) = 0 \quad (\text{A.34})$$

with initial condition:

$$\Re(0) = 1. \quad (\text{A.35})$$

So  $\Re(p)$  can be represented by expression:

$$\Re(p) = \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{\Gamma^2 \left( k + \frac{3}{2} \right) p^k}{k! (k+2)!}. \quad (\text{A.36})$$

It means that while computation function  $\Re(p)$  can be found using the following algorithm:

$$\Re(p) = \sum_{k=0}^K \Re_k(p), \quad (\text{A.37})$$

where

$$\Re_0(p) = 1, \quad \Re_k(p) = \frac{\left( k + \frac{1}{2} \right)^2}{k(k+2)} p \Re_{k-1}(p), \quad (\text{A.38})$$

and  $K$  – the number of summary component, which already gives the necessary accuracy  $\delta$ :

$$\frac{\Re_K(p)}{\Re(p)} \leq \delta. \quad (\text{A.39})$$

The series (A.36) is theoretically absolutely coinciding one in the range of argument

$$0 \leq p < 1 \quad (\text{A.40})$$

with logarithmical singularity near  $p=1$ .

But while computation the apparatus error leads to non-coincidence of (A.36) when  $p \rightarrow 1$ . The following asymptotic series can be used there:

$$\Re(p) = \frac{32}{\pi^2} \sum_{k=0}^{\infty} \frac{\Gamma^2\left(k + \frac{3}{2}\right)}{(k!)^2} \left[ \ln \frac{1}{1-p} + \sqrt{\pi} - 3 + \sum_{n=1}^k \frac{1}{n\left(n + \frac{1}{2}\right)} \right] (1-p)^k, \quad (\text{A.41})$$

or

$$\Re(p) = \sum_{k=0}^{\infty} C_k(p) \left( \ln \frac{1}{1-p} + R_k \right), \quad (\text{A.42})$$

where

$$\begin{aligned} C_0(p) &= \frac{8}{\pi}, & C_k(p) &= \frac{\left(k + \frac{1}{2}\right)^2}{k^2} p C_{k-1}(p), \\ R_0 &= \sqrt{\pi} - 3, & R_k &= R_{k-1} + \frac{1}{k\left(k + \frac{1}{2}\right)}. \end{aligned} \quad (\text{A.43})$$

## References

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<sup>2</sup>Dnetrovcki, YU.N., and Koctomarov D.P., *Plasma Mathematical Modeling*, 2<sup>nd</sup> ed., Nayka, Moscow, 1993, Chaps. 3.