

Low frequency mode transitions using dynamical systems modeling

IEPC-2017-540

*Presented at the 35th International Electric Propulsion Conference
Georgia Institute of Technology – Atlanta, Georgia – USA
October 8–12, 2017*

Daniel Eckhardt*

NRC Postdoctoral Fellow, AFRL/RQRS, USA

Justin Koo†

AFRL/RQRS, USA

Michelle Scharfe‡

ERC Incorporated, Huntsville, AL, 35806-3200, USA

Mode transitions in Hall-Effect thrusters (HETs) are of particular interest, owing to the constraints they put on the operation and lifetime of the thruster. In particular, changes in the fundamental characteristics of the breathing mode, which is characterized by large current discharges, are of interest because of the insight they provide to global thruster operation and performance. It has been found that the dynamics of this mode can be captured to a certain extent using the predator-prey model,¹ which is based on continuity equations for the neutral and ion density in a zero dimensional ionization box. We consider the stability criteria of this dynamical system in order to demonstrate the inability of the classical system to capture mode transitions through some form of bifurcation. In addition, by considering the dynamics as a coupled system of slow and fast manifolds, we propose a formulation capable of expressing this bifurcation.

Nomenclature

n	=	Ion density
N	=	Neutral density
N_{int}	=	Initial neutral density
U_{int}	=	Initial neutral velocity
V_i	=	Ion velocity
V_n	=	Neutral velocity
V_{ex}	=	Electron velocity
ν_i	=	Net ionization frequency
ν_{iw}	=	Ion-wall collisional frequency
m_e	=	Mass of electrons
T_e	=	Electron temperature
P_{ew}	=	Electron power losses to the wall
ν_e	=	Effective electron collision frequency
ξ	=	Ionization rate
L	=	Characteristic length of ionization channel
L_{ch}	=	Channel width

*daniel.eckhardt.2.ctr@us.af.mil

†justin.koo@us.af.mil

‡michelle.scharfe.ctr@us.af.mil

I. Introduction

The dynamic behavior of Hall thrusters displays a rich variety of oscillatory behavior. The most commonly recognized demonstration of this is the characteristic oscillatory behavior of the discharge current, known as discharge current oscillations, in the 15-40 kHz range. It is well known that changes in the applied magnetic field can dramatically affect the behavior of the discharge current oscillations (mean and standard deviation). Nevertheless, the discharge current continues to qualitatively display the classical breathing mode oscillations regardless of the magnetic field strength. For example, discharge current traces for the H6 HET³⁴⁵ clearly show two distinct breathing mode oscillations, commonly described as local mode and global mode operation, which are highly recognizable as being from the same family of discharge current oscillations as seen below.

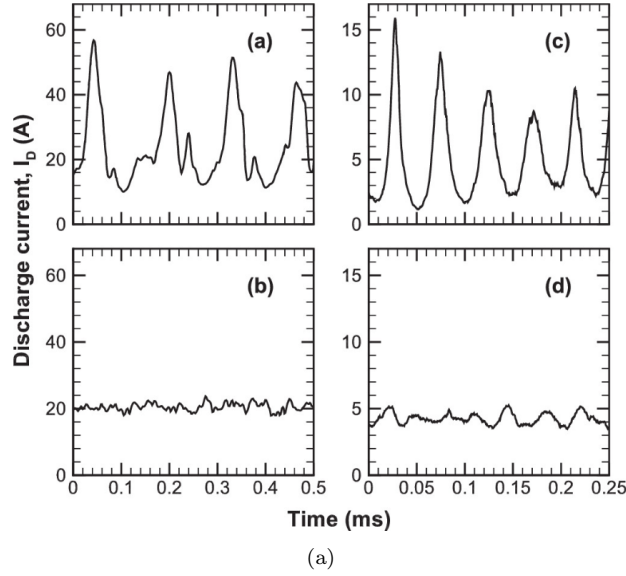


Figure 1: Experimental H6 HET discharge current traces (left) and numerical SPT-100 HET discharge current traces (right) showing two different breathing modes.⁵

These discharge current oscillations, also referred to as *breathing mode oscillations*, have been studied in detail and the most widely accept analysis of this phenomena was first proposed by Fife.¹ This analysis was based on a competition between the timescales of the neutral ionization process and neutral refill time. This process, most famously applied to the biological interaction between populations of foxes (predator - y) and hares (prey - x), is also known as a predator-prey relationship, can be described mathematically by the Lotka-Volterra equations^{6,7} as shown below:

$$\begin{aligned}\frac{dx}{dt} &= \alpha x - \beta xy, \\ \frac{dy}{dt} &= -\gamma y + \delta xy.\end{aligned}\tag{1}$$

In the phase space described by x - y , nontrivial solutions of the Lotka-Volterra equations result in characteristic limit cycle oscillations of closed orbits around an equilibrium point as show in figure 2 below.

While the Lotka-Volterra equations can be successfully applied to an individual thruster mode, this paper seeks to understand whether this form can be extended to capture the dynamics of a mode change. Two approaches were studied in this investigation. The first was a theoretical investigation of the Lotka-Volterra equations to understand whether the classical form of these equations can support multiple non-zero equilibria - a necessary condition to support mode transitions. The conclusion of this investigation led to a second more physically based investigation to look into additional physics terms (representing high frequency behavior not captured by the slower Lotka-Volterra dynamics) which would enable multiple non-zero equilibria. This second investigation studies the theoretical implications of these extra terms and demonstrates their ability

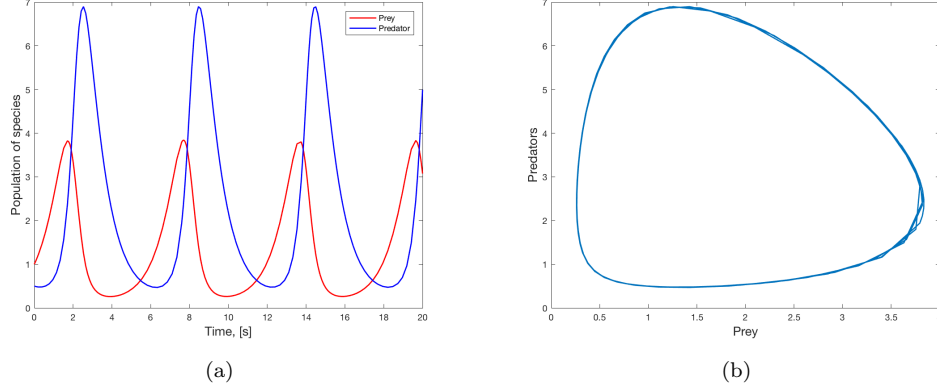


Figure 2: Typical oscillations (a) and phase portrait (b) of predator prey model with arbitrary parameters.

to support bifurcated solutions within the context of a single equation system describing the dynamics of the thruster system.

II. Predator-prey model

For complex nonlinear dynamical systems, we can study and quantify their dynamics by considering the behavior of the linearized system about its equilibrium points, as guaranteed by the Hartman-Grobman theorem.⁸ We can study the breathing mode by generalizing the flow to a problem of the form

$$\partial_t \mathbf{x} = \mathbf{f}(\mathbf{x}, t), \quad (2)$$

where $\mathbf{x} \in \mathbf{X} \subseteq \mathbb{C}$, $\mathbf{f} \in \mathbf{X}^*$ and forward complete, and $\mathbf{f} \in C(\mathbb{R}^n)$, or at least sufficiently smooth. When $x_e \in \mathbb{R}$ is an equilibrium point, i.e. $\mathbf{f}(\mathbf{x}_e) = 0$ and $J(x_e) = [\partial f_i(x_e)/\partial x_j]$ of $\mathbf{f}$ at x_e , the linearization of $\mathbf{f}$ at x_e , has no eigenvalues of the form $Re(\lambda_j) = 0$, then $\exists \mathcal{N}(x_e)$ and a homeomorphism $h : \mathcal{N} \rightarrow \mathbb{R}^n$ s.t. $h(x_e) = 0$ and s.t. in $\mathcal{N}$; $d\mathbf{x}/dt = \mathbf{f}(\mathbf{x})$ is topologically conjugate by a continuous map. By Routh-Hurwitz stability criterion,⁸ we also know that a Hopf Bifurcation arises when the two conjugate eigenvalues cross the imaginary axis because of a variation of the system parameters. Hence a necessary condition is the existence of a conjugate pair of eigenvalues. This gives us one way to consider the breathing mode; i.e. by considering existence of limit cycles and or attractors, as well as mode hops between them through bifurcation. Next, we apply this analysis to the predator-prey description from¹ obtained from considering the continuity equations and moving to a 0-D formulation by the assumption that $\partial_x \rightarrow \frac{1}{L}$

$$\begin{aligned} \frac{dN}{dt} &= -nN\xi(T_e) + \frac{NV_n}{L}, \\ \frac{dn}{dt} &= +nN\xi(T_e) - \frac{nV_i}{L}, \end{aligned} \quad (3)$$

Interestingly, from these equations alone one can identify four unique timescales,

$$\begin{aligned} \tau_1 &= \frac{1}{-n\xi(T_e)}, \\ \tau_2 &= \frac{L}{V_n}, \\ \tau_3 &= \frac{1}{N\xi(T_e)}, \\ \tau_4 &= \frac{L}{V_i}, \end{aligned} \quad (4)$$

hinting at a need to consider separation of manifolds on which this simple system is acting. The resulting behavior of this system can be seen in Fig. 3.

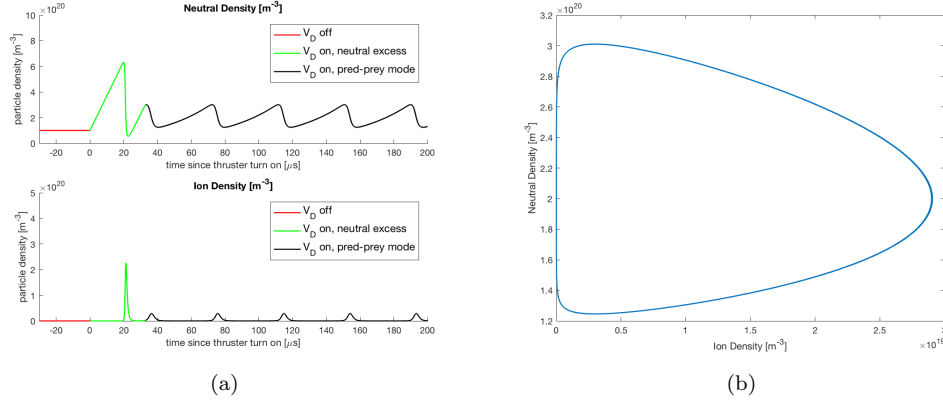


Figure 3: Thruster operation under classical Lotka-Volterra model

If we consider dynamical systems theory and linearize (3) about its equilibrium points, we get that the trivial equilibrium point is hyperbolic but its eigenvalues fail to satisfy Routh-Hurwitz and hence we don't expect a bifurcation to occur at that point. On the other hand, the nontrivial equilibrium point gives eigenvalues $\lambda_{1,2} = \pm i\sqrt{\frac{V_n V_i}{L^2}}$, which would give the necessary condition for a bifurcation using Routh-Hurwitz, but since it is a nonhyperbolic equilibrium, we consider the center manifold theorem.⁸ However, this too is a stable equilibrium point and thus fails to allow for the system to undergo a bifurcation. For example consider the candidate Lyapunov function $\varphi = \frac{1}{2}(c_1 n^2 + c_2 N^2)$, we see that $\dot{\varphi} = (c_1 - c_2)nN\xi(T_e) - c_1 n^2 V_i + c_2 N^2 V_n$. A trivial choice of $c_1 = c_2$ tells us that this equilibrium point is stable considering the scaling involved with n, N, V_n, V_i . In addition, this system only presents one attractor, and thus, it is possible to claim that if the fundamental nature of breathing mode transitions is a result of the underlying dynamical system moving from one attractor to another, this bifurcation cannot exist for the classical Lotka-Volterra form of (3) and more terms or equations would need to be added to the system to demonstrate this behavior. There have been various efforts to reconcile this fact with rigorous mathematics, for example⁹ and².

A few things become clear from these and other analyses – either one has to completely find a new description of the model, or perhaps introduce new terms to (3), in order to make it feasible for it capture the transitions of the breathing mode. One straightforward way we can consider is the addition of an abstract forcing term that kicks the system out of this stable limit cycle onto another stable limit cycle. In essence, this forcing term is capturing dynamics that don't interact directly with the slow manifold on which the basic model (3) is operating, until some criteria is met. For example, we can say that since (3) at most works on the time scale of neutrals, there could be a coupled *faster* manifold of the charged species, that works to change the ion density, in a way that forces (3) to move to another stable attractor or limit cycle in this case. This can be abstractly captured as

$$\begin{aligned}\frac{dn}{dt} &= nN\xi(T_e) - \frac{nV_i}{L} - \mu n \\ \frac{dN}{dt} &= -nN\xi(T_e) + \frac{NV_n}{L} + \mu n,\end{aligned}\tag{5}$$

where μ is the forcing term that abstractly couples the *faster* manifold dynamics to (3). The exact form of μ can be determined by considering other analyses of the breathing mode, but here we demonstrate that it is sufficient to show that the expected behavior is reproduced. For example here we choose μ as a bifurcation parameter between two local manifolds.

In Fig. 4 we see the behavior of a thruster under this formulation. Clearly the system is in a stable limit cycle until the addition of finite μ causes the system to undergo a transition to a spiral centered on different equilibrium point, thus demonstrating a mode transition between limit cycle oscillations.

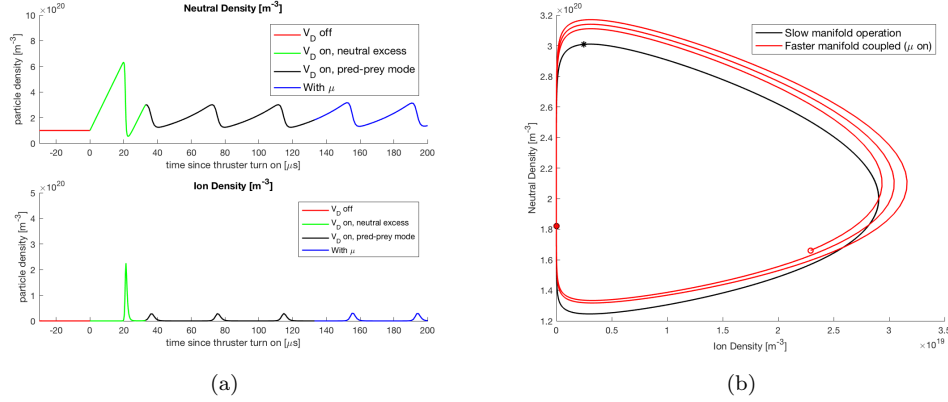


Figure 4: In the phase space plot in (b) the black line represents behavior under classical LV operation and the red (maybe blue?) line represents the new behavior under the integrated influence of the unresolved faster manifold (i.e. $\mu \neq 0$ at $t > 130 \mu\text{s}$).

1. Perturbation analysis of forced model

We can perform an analysis of (5), by considering linear perturbations of the form $n(x, t) = n_0(x) + n'(x, t)$ where $n'(x, t) = n'(x)e^{-i\omega t}$. Substituting into (5), we get that the equilibrium densities are

$$N_0 = \frac{V_i}{L\xi} + \frac{\mu}{\xi} \quad (6)$$

$$n_0 = \frac{V_n}{L\xi} \left(1 - \frac{\mu}{N_0\xi}\right)^{-1} \quad (7)$$

The first order perturbation is given as

$$\begin{aligned} \frac{dn'}{dt} &= n'N_0\xi + n_0N'\xi - n'\frac{V_i}{L} - n'\mu \\ \frac{dN'}{dt} &= -n'N_0\xi - n_0N'\xi + N'\frac{V_n}{L} + n'\mu, \end{aligned}$$

Using (6) and (7) together with the approximation that $\xi N_0 \gg \mu$, we get that

$$\frac{d^2n'}{dt^2} = -N_0n'\xi\frac{V_n}{L}, \quad (8)$$

which gives a frequency of $\omega^2 = -N_0\xi(T_e)\frac{V_n}{L}$, an oscillation frequency slightly differs from that quoted in other sources. Note that the physics contained in the forcing parameter μ is hidden in the above equation for simplicity. It is also useful to note that in fact, since the system doesn't quite behave the same way on a global manifold, and it seems more likely that one would need to consider formulations that take into account the existence of *slow* and *fast* manifolds.

A. A more geometrically correct model

Both models given in equations (3) and (5) lack a geometric interpretation, which despite still considering a 0-d case, is still quite informative. We can consider an approach similar as was attempted in Hara, et. al⁴ but with a term that we feel should be added to make the equations consistent.

So from the above configuration, we can get from the continuity equations that

$$\begin{aligned} \frac{dn}{dt} + \frac{nV_i}{L_{ch}} + \frac{2nV_{iw}}{R} &= nN\xi(T_e) \\ \frac{dN}{dt} + \frac{(N - N_{int})V_n}{L_{ch}} - \frac{2nV_{iw}}{R} &= -nN\xi(T_e) \end{aligned} \quad (9)$$

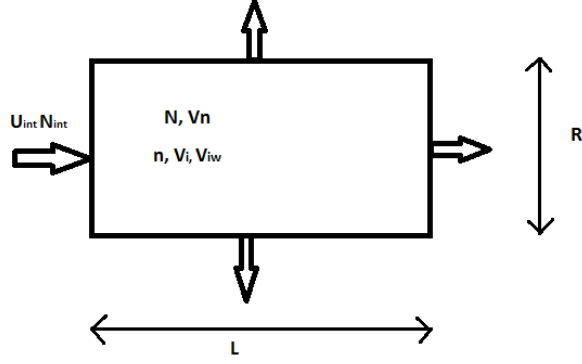


Figure 5: Geometrical representation of ionization box

L_{ch} is the channel length which we shall now call L , R is the channel depth, and we take U_{int} as V_n . As expected, we can see in Fig. 6 the classical breathing mode behavior of the thruster with this model.

For the sake of completeness, we can now perform a perturbation analysis of this new model (15) in order to compare with work in Hara, et. al.⁴ We again take $n = n_0 + n'$ and get that

$$\begin{aligned} \frac{dn'}{dt} &= -\frac{(n_0 + n')V_i}{L_{ch}} - \frac{2(n_0 + n')V_{iw}}{R} + (n_0 + n')(N_0 + N')\xi \\ \frac{dN'}{dt} &= -\frac{((N_0 + N') - N_{int})V_n}{L_{ch}} + \frac{2(n_0 + n')V_{iw}}{R} - (n_0 + n')(N_0 + N')\xi, \end{aligned}$$

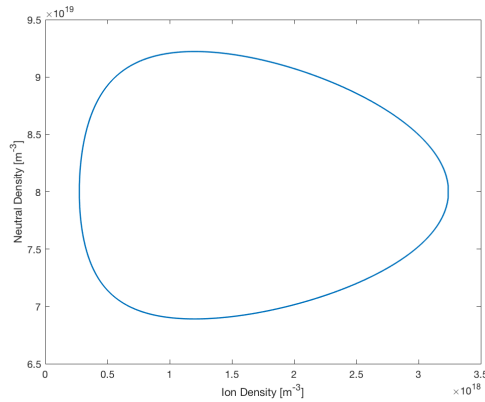


Figure 6: Thruster running with new geometrically correct model

which gives the equilibrium densities as

$$N_0 = \frac{V_i}{L\xi} + \frac{2V_{iw}}{R\xi}, \quad (10)$$

$$n_0 = \frac{V_n}{V_i}(N_{int} - N_0), \quad (11)$$

as opposed to $n_0 = (N_{int} - N_0) \frac{V_n}{V_i} \cdot L_{ch} \left(1 + \frac{2V_{iw}}{V_i} \frac{L_{ch}}{R}\right)^{-1}$ in⁴ which contrasts with (11) which has no direct dependence on the channel length. We still have the condition that $N_{int} > N_0$. We also have that the ion acceleration is also dependent on the *ion acoustic velocity acceleration*, thus the minimum T_e varies. From the 1st order perturbations, we get that

$$\frac{\partial n'}{\partial t} = n_0 N' \xi, \quad (12)$$

$$\frac{\partial N'}{\partial t} = -\frac{n' V_i}{L} - \left(\frac{V_n}{L} + n_0 \xi\right) N'. \quad (13)$$

Like in Hara, et. al,⁴ we can describe the perturbed system as

$$\begin{bmatrix} -i\omega & -n_0 \xi \\ \frac{V_i}{L} & -i\omega - \left(\frac{V_n}{L} + n_0 \xi\right) \end{bmatrix} \cdot \mathbf{f} = \mathbf{0}, \quad (14)$$

where $\mathbf{f} = [n', N']^T$. We can find the wave frequency by solving for ω as an eigenvalue of above matrix, which gives us that $-\omega^2 + i\omega\gamma + \frac{V_i n_0 \xi}{L} = 0$ where $\gamma = -\left(\frac{V_n}{L} + n_0 \xi\right)$, which we can rewrite using (11) as $\gamma = -\frac{V_n}{L} \left(1 + r\xi \frac{(N_{int} - N_0)}{N_0 \xi - 2V_{iw}}\right)$. So we have that $\omega = \frac{i\gamma}{2} \mp \frac{1}{2} \sqrt{-\gamma^2 + 4 \frac{n_0 \xi V_i}{L}}$. We can split this into a real frequency and a growth rate: $\omega = \omega_R + \gamma$. We have stability if $\gamma < 0$, which is trivially true from our choice of γ . Based on this and eigenvalue analysis of the original system, we can conclude that even this geometrically improved system cannot support breathing mode transitions.

1. Forcing the improved geometric model

To capture our assertion from the previous section of the effects of a faster manifold on the slower manifold through the parameter μ , we can add to this system to recover:

$$\begin{aligned} \frac{dn}{dt} + \frac{nV_i}{L_{ch}} + \frac{2nV_{iw}}{R} &= nN\xi(T_e) - \mu n \\ \frac{dN}{dt} + \frac{(N - N_{int})V_n}{L_{ch}} - \frac{2nV_{iw}}{R} &= -nN\xi(T_e) + \mu \end{aligned} \quad (15)$$

With the above addition of this forcing term representing an interaction with the faster manifold, we see a clear mode hope to the new, decaying limit cycle in Fig. 7. Continuing to run the simulation beyond 1500 μs leads to a limit cycle oscillation which is clearly different from the limit cycle oscillation in the no-forced period of $50\mu s < t < 500\mu s$ as shown in Fig. 8.

III. Conclusions and Future Work

The primary motivation for this paper was to address the suitability of the classical Lotka-Volterra equation system (as adapted for O-D ionization processes in HETs) to capture mode transitions in the breathing mode oscillations. Analysis of the linearized system around its equilibrium points indicated that no Hopf bifurcations are possible, leading to the conclusion that the classical Lotka-Volterra equation system alone cannot capture mode transitions. To extend the equation system, we used the ansatz of a simple forcing term, ostensibly to represent the integrated effects of interactions with unresolved higher frequency behavior. Switching this term on enabled the system to jump from one limit cycle to a different limit cycle, demonstrating the theoretical capability to capture mode transitions. Finally, in a parallel track, this paper attempted to add slightly more rigor to collapse of the continuity equations to a 0-D formulation but demonstrated,

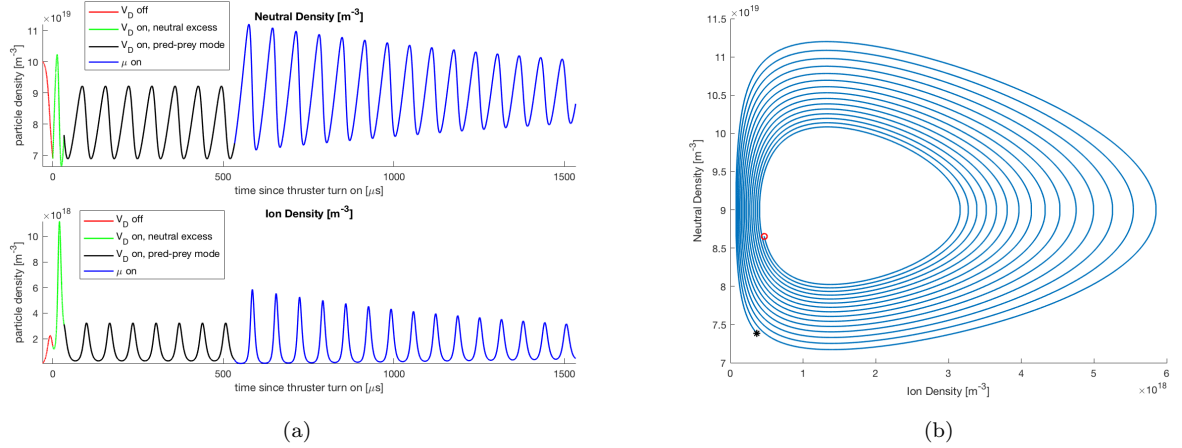


Figure 7: Thruster running with new geometrically correct model; Phase plot on RHS represents behavior at $t > 100 \mu s$ with introduction of forcing term.

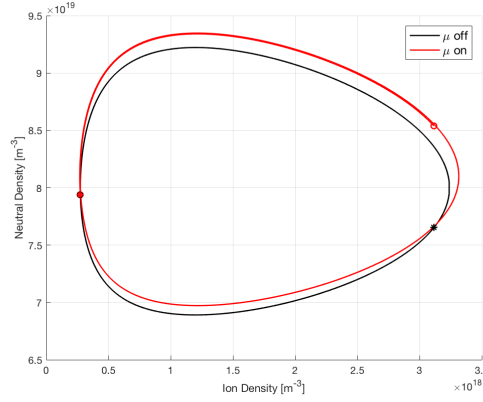


Figure 8: Limit cycle oscillations running with new geometrically correct model

yet again, that even this more complicated 0-D formulation without additional forcing terms is incapable of capturing mode transitions.

This analysis hints that addition of additional independent variables may provide a physics-base approach to achieving the results of our simple forcing term ansatz. Initial investigations were performed into the addition of time-dependent ion momentum and energy equations and steady state electron momentum to the ion and neutral continuity equations studied in this paper. Future work consists of continuing this investigation to see if the new independent parameters (V_i and T_e) can play a forcing role similar to μ as seen in the Appendix.

Acknowledgement

This work was supported in part by NRC grant FA9550-17-D-0001 and AFSOR grant FA9550-17RQCOR465.

References

- ¹Fife, J. M., “Hybrid-PIC modeling and electrostatic probe survey of Hall thrusters,” .
- ²Barral, S. and Ahedo, E., “Theoretical study of the breathing mode in Hall thrusters,” in “Proc. 42nd AIAA Joint Propulsion Conference,” Vol. 5172, 2006.
- ³Sekerak, M. J., *Plasma oscillations and operational modes in Hall effect thrusters*, Ph.D. thesis, National Aeronautics and Space Administration, 2014.

⁴Hara, K., Sekerak, M. J., Boyd, I. D., and Gallimore, A. D., “Perturbation analysis of ionization oscillations in Hall effect thrusters,” *Physics of Plasmas*, Vol. 21, No. 12, 2014, p. 122103.

⁵Hara, K., Sekerak, M. J., Boyd, I. D., and Gallimore, A. D., “Mode transition of a Hall thruster discharge plasma,” *Journal of Applied Physics*, Vol. 115, No. 20, 2014, p. 203304.

⁶Lotka, A. J., “Contribution to the theory of periodic reactions,” *The Journal of Physical Chemistry*, Vol. 14, No. 3, 1910, pp. 271–274.

⁷Volterra, V., “Variations and fluctuations of the number of individuals in animal species living together,” *ICES Journal of Marine Science*, Vol. 3, No. 1, 1928, pp. 3–51.

⁸Perko, L., *Differential equations and dynamical systems*, Vol. 7, Springer Science & Business Media, 2013.

⁹Barral, S. and Peradzynski, Z., “A new breath for the breathing mode,” in “31st International Electric Propulsion Conference, University of Michigan, Ann Arbor, Michigan, USA,” , 2009, pp. 20–24.

Appendix

Consider 1-D ion momentum, energy and steady state electron momentum equations, (16, 17 and 18 respectively). We can investigate whether linear perturbations applied to this model decay as follows:

$$\frac{\partial}{\partial t}(m_i n V_i) + \frac{\partial}{\partial x}(m_i n V_i^2) = -ne \frac{\partial}{\partial x} \phi + \beta_i n m_i N V_n - \nu_{iw} m_i n V_i, \quad (16)$$

$$\frac{\partial}{\partial t} \left(\frac{3}{2} n T_e \right) + \frac{\partial}{\partial x} \left(\frac{5}{2} n v_{ex} T_e \right) = n e v_{ex} \frac{\partial}{\partial x} \phi - \beta_i n N \epsilon_i - n P_{ew}, \quad (17)$$

and

$$\frac{\partial}{\partial x} (n T_e) = n e \frac{\partial}{\partial x} \phi - m_e n v_{ex} \nu_e. \quad (18)$$

This gives us that

$$\frac{V_{i,0}^2}{L} = -\frac{e}{m_i} \frac{\phi}{L} + N_0 \xi_0 V_n - \nu_{iw} V_{i,0}, \quad (19)$$

or if we take $\nu_{iw} \sim \frac{2V_{iw}}{R}$ and see that

$$V_{i,0} = \left(\frac{e}{m_i} \frac{\phi}{L} + \frac{2V_n V_{iw}}{R} \right) \left(\frac{2V_{iw}}{R} + \frac{V_n}{L} \right)^{-1}. \quad (20)$$

Similarly we get that for temperature,

$$\frac{3}{2} T_{e,0} = \left(\frac{V_{ex}}{L} \right)^{-1} [m_e n_0 V_{ex}^2 \nu_{e,0} - \xi_0 n_0 N_0 \epsilon_{i,0} - n_0 P_{ew}]. \quad (21)$$

So we get that for

$$\mathbf{f} = \begin{bmatrix} n'_i & n'_n & V'_i & T'_e \end{bmatrix},$$

we have that

$$\begin{bmatrix} -i\omega & -n_0 \xi_0 & \frac{n_0}{L} & -n_0 N_0 \\ \frac{V_{i,0}}{L} & (-i\omega + \frac{V_n}{L} + n_0 \xi_0) & 0 & n_0 N_0 \\ \aleph_1 & -n_0 \xi_0 V_n & \aleph_2 & -n_0 N_0 \\ \aleph_3 & \aleph_4 & 0 & \aleph_5 \end{bmatrix} \cdot \mathbf{f} = 0 \quad (22)$$

where

$$\begin{aligned}
\aleph_1 &= -i\omega V_{i,0} + \frac{e\phi}{m_i L} - \frac{V_n V_{i,0}}{L} + \frac{2V_{iw}}{R}(V_{i,0} - V_n) \\
\aleph_2 &= -i\omega n_0 + 2n_0 \left(\frac{V_{i,0}}{L} + \frac{V_{i,w}}{R} \right) \\
\aleph_3 &= -\frac{3}{2}i\omega T_{e,0} + \frac{3}{2} \frac{V_{ex}}{L} T_{e,0} - m_e V_{ex}^2 \nu_{e,0} + 2n_0 \frac{V_{iw}}{R} + \frac{(N_{int} - N_0)V_n}{L} + P_{ew} \\
\aleph_4 &= \frac{(N_{int} - N_0)V_n}{N_0 L} + \frac{2n_0 V_{iw}}{N_0 R} \\
\aleph_5 &= -\frac{3}{2}i\omega n_0 + \frac{3}{2} \frac{V_{ex}}{L} n_0 - m_e V_{ex}^2 \nu'_e n_0 + n_0 N_0 \epsilon_{i,0} + \frac{(N_{int} - N_0)V_n \epsilon'_i}{L} + 2n_0 \frac{V_{iw}}{R}.
\end{aligned}$$

Using this fuller analogous model to (14), we can investigate whether perturbations decay similar to Hara, et. al,⁴ and we can also introduce the abstract parameter μ to account for the physics that we argued earlier.